THE ZAK TRANSFORM ON CERTAIN LOCALLY COMPACT GROUPS

GITTA KUTYNIOK

ABSTRACT. For a class of locally compact groups, which includes all connected and simply connected 2-step nilpotent Lie groups whose Lie algebra admits a basis with respect to which the structure constants are rational, we introduce a definition of the Zak transform. We prove that this Zak transform is quasi-periodic. Further, we show that it is a Hilbert space isomorphism when the range functions are restricted to a fundamental domain. Finally, we study several examples of locally compact groups with respect to properties of their Zak transform.

1. INTRODUCTION

1.1. **History.** The Zak transform on $\mathbb{R}$ was introduced in 1950 by Gelfand [Gel50] and it was rediscovered by Weil [Wei64] and independently by Zak [Zak67] in 1967 who used it to construct a quantum mechanical representation for the description of the motion of a Bloch electron in the presence of a magnetic or electric field. For $f \in L^2(\mathbb{R})$, the *Zak transform* is the function $Zf : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$Zf(x,y) = \sum_{k=-\infty}^{\infty} f(x+k)e^{2\pi iyk}.$$

The Zak transform later became a major tool in the analysis of Gabor systems, since it turned out to be highly efficient, for example, for integer oversampling. A review of the theory and of applications to signal analysis can be found in the survey article of Janssen [Jan88].

Weil [Wei64] already introduced the Zak transform in the general context of locally compact abelian groups. In the last 10 years this has been rediscovered in engineering for such groups as $\mathbb{Z}$ or finite cyclic groups [AGT92, Hei89]. But so far there are only a few papers dealing with the Zak transform on general locally compact abelian groups ([FS98, Chapter 6] and [KK98]).

1.2. **Outline.** In the present paper we generalize the notion of Zak transform to a large class of locally compact groups including all connected and simply connected 2-step nilpotent Lie groups whose Lie algebra admits a basis with

1

respect to which the structure constants are rational, study some of their properties and present several examples.

This paper is organized as follows.

In the second section we introduce some notation, recall a few definitions and a simple basic result which will be used in the sequel.

The purpose of the third section is to define the Zak transform on certain locally compact groups by ensuring that many of the main properties of the classical Zak transform remain true. We also state some general hypotheses for this section and the following two sections.

In Section 4 we check whether the general definition reduces to the classical definition for $G = \mathbb{R}$. Further, we prove some quasi-periodicity relation for the Zak transform and show that the range functions of the Zak transform are uniquely determined by their values on a fundamental domain.

The fifth section deals with the problem under which conditions the Zak transform is an isometry and in particular a Hilbert space isomorphism, when the range functions are restricted to a fundamental domain.

In the last section we discuss several examples of groups like locally compact abelian groups and connected and simply connected 2-step nilpotent Lie groups whose Lie algebra admits a basis with respect to which the structure constants are rational and examine properties of their Zak transform.

2. Preliminaries and Notation

Let G be a locally compact group with neutral element denoted by e. For $K \subseteq G$, $K < G$ denotes the fact that K is a subgroup of G. A subgroup K of G will be called a *uniform lattice*, if it is discrete and cocompact. A *fundamental domain* for K in G is a measurable cross section S_K, that means a measurable set $S_K \subseteq G$ such that every $x \in G$ can be uniquely written in the form $x = ks$ where $k \in K$ and $s \in S_K$. The existence of a fundamental domain for K is guaranteed by the following lemma, which is [KK98, Lemma 2]. It is stated in [KK98] only for locally compact abelian groups, but it also holds for non-abelian locally compact groups using the same proof.

Lemma 2.1. *Let G be a locally compact group and let K be a uniform lattice in G. Then there exists a relatively compact fundamental domain for K in G.*

Let $L^2(G)$ denote the space of square-integrable functions on G with respect to some left Haar measure on G. The group of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms, endowed with multiplication given by composition, is denoted by $\mathrm{Aut}(G)$. Let G and H be two locally compact groups and $\tau : G \to \mathrm{Aut}(H)$ be a homomorphism such that the mapping $(x, h) \mapsto \tau_x(h)$ is a continuous mapping of $G \times H$ onto H. Then the cartesian product $G \times H$ endowed with the multiplication

$$(x_1, h_1)(x_2, h_2) = (x_1 x_2, h_1 \tau_{x_1}(h_2))$$

and with the product topology is called a *semidirect product* of G and H and is denoted by $G \ltimes_\tau H$.

Now let G be a locally compact abelian group. The dual group of G is always denoted by $\widehat{G}$ with unit element 1_G (or just 1). Let K be a subgroup of G. Then the subgroup $A(K, \widehat{G}) = \{\omega \in \widehat{G} : \omega(k) = 1 \text{ for all } k \in K\}$ is called the *annihilator of K in $\widehat{G}$*. For a uniform lattice K in G, the subgroup $A(K, \widehat{G})$ is a uniform lattice in $\widehat{G}$, since $A(K, \widehat{G}) = \widehat{G/K}$ and $\widehat{G}/A(K, \widehat{G}) = \widehat{K}$ ([HR63, Lemma 24.5]) and since the dual of a compact abelian group is discrete and vice versa. As a general reference to duality theory of locally compact abelian groups we mention [HR63].

The *Heisenberg group associated with G*, $H(G)$, is the semidirect product $G \ltimes_\tau (\widehat{G} \times \mathbb{T})$, where

$$\tau : G \to \text{Aut}(\widehat{G} \times \mathbb{T}), \quad \tau_x(\omega, z) = (\omega, z\omega(x)).$$

In the following we will consider the so-called *Schrödinger representation*, which is the irreducible unitary representation of $H(G)$ on $L^2(G)$ defined by

$$(\rho_G(x, \omega, z)f)(t) = z\,\omega(t)f(xt).$$

This is the natural generalization of the Schrödinger representation of $H(\mathbb{R})$ dealt with in Gabor analysis. For further information concerning Heisenberg groups we refer to [Fol89].

3. Definition of the Zak transform

Let G be a locally compact abelian group. The Zak transform Zf associated with a uniform lattice K in G of $f \in L^2(G)$ is defined on $G \times \widehat{G}$ by

$$Zf(x, \omega) = \sum_{k \in K} f(xk)\omega(k).$$

Let S_K and Ω_K be fundamental domains for K in G and for $A(K, \widehat{G})$ in $\widehat{G}$, respectively. Since the Zak transform satisfies

$$Zf(xk, \omega\gamma) = \overline{\omega(k)}Zf(x, \omega) \text{ for all } (x, \omega) \in G \times \widehat{G}, (k, \gamma) \in K \times A(K, \widehat{G}),$$

the so-called quasi-periodicity relation, the function Zf is uniquely determined by its values on $S_K \times \Omega_K$. By [KK98, Lemma 3], the mapping $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is an isometry. Moreover, if G is second countable, it can be shown that it is even a Hilbert space isomorphism ([Ku00, Theorem 3.1.7]).

Now we are interested in a generalization of this definition of the Zak transform to some non-abelian locally compact groups. This should be achieved in such a way that the above mentioned properties remain true. Notice first that we may rewrite the Zak transform in the following way

$$Zf(x, \omega) = \sum_{k \in K} (\rho_G(x, \omega, 1)f)(k).$$

The map ρ_G is a representation of the Heisenberg group associated with G, $H(G)$, on $L^2(G)$. Note that we have $H(G) = G \ltimes_\tau (\widehat{G} \times \mathbb{T})$, where $\tau : G \to \text{Aut}(\widehat{G} \times \mathbb{T})$ is given by $\tau_x(\omega, z) = (\omega, z\omega(x))$. In addition, we have $Z(H(G)) =$

$\mathbb{T}$. Denoting $S_{(e,1)}$ the stabilizer of $(e,1) \in G \times \mathbb{Z}$ $(= (\widehat{G} \times \mathbb{T})\hat{\,})$, it is easily checked that $S_{(e,1)} = \{e\}$. It is well-known that the Schrödinger representation ρ_G is equivalent to the representation induced by the character $(e,1) \in G \times \mathbb{Z}$.

Now let G be a locally compact group and let K be a uniform lattice in G. Moreover, let L and Z be locally compact abelian groups and let $\tau : G \to \mathrm{Aut}(L \times Z)$ be an action such that $Z(G \ltimes_\tau (L \times Z)) = Z$. Then the group $G \ltimes_\tau (L \times Z)$ will play a similar role as the Heisenberg group. Further we suppose that there exists some $\chi \in \widehat{Z}$ such that $S_{(1,\chi)} = \{e\}$. Then the induced representation

$$\rho := \mathrm{ind}_{\{e\}\ltimes_\tau(L\times Z)}^{G\ltimes_\tau(L\times Z)}(1_{\widehat{L}}, \chi) : G \ltimes_\tau (L \times Z) \to \mathcal{U}(L^2(G))$$

replaces the Schrödinger representation.

Thus, throughout the remainder of this section and Sections 4 and 5, let G be a locally compact group and let K be a uniform lattice in G. By Lemma 2.1, there exists a relatively compact fundamental domain for K in G which we will denote by S_K in the following. In addition, suppose that

(I) there exist locally compact abelian groups L and Z and some action $\tau = (\tau^{(1)}, \tau^{(2)}) : G \to \mathrm{Aut}(L \times Z)$ such that $Z(G \ltimes_\tau (L \times Z)) = Z$ and such that the map $y \mapsto \tau_k^{(1)}(y, e)$, $L \to L$, is an isomorphism for all $k \in K$,

(II) there exists some $\chi \in \widehat{Z}$ such that the map

$$x \mapsto \chi(\tau_x^{(2)}(\cdot, e)), \quad G \to \widehat{L},$$

is injective.

For the definition of the representation, which shall generalize the Schrödinger representation, we need the following lemma.

Lemma 3.1. *We have*

$$S_{(1,\chi)} = \{e\}.$$

Proof. Using the fact that $Z(G \ltimes_\tau (L \times Z)) = Z$, it is easily checked that

$$S_{(1,\chi)} = \{x \in G : \chi(\tau_x^{(2)}(y,e)) = 1 \quad \text{for all } y \in L\}.$$

Hence the claim follows from (II) of the preceding assumptions. $\qquad\square$

From now on let $\rho : G \ltimes_\tau (L \times Z) \to \mathcal{U}(L^2(G))$ be the unitary representation defined by

$$\rho := \mathrm{ind}_{\{e\}\ltimes_\tau(L\times Z)}^{G\ltimes_\tau(L\times Z)}(1_{\widehat{L}}, \chi).$$

By [Fol95, Section 6.1], we obtain

$$(\rho(x,y,z)f)(t) = \chi(z)\chi(\tau_t^{(2)}(y,e))f(tx)$$

for all $(x,y,z) \in G \ltimes_\tau (L \times Z)$, $t \in G$ and $f \in L^2(G)$.

Definition 3.2. The *Zak transform* Zf *associated with* K *(and* L, Z, τ *and* χ*) of* $f \in L^2(G)$ *is defined on* $G \times L$ *by*

$$Zf(x, y) = \sum_{k \in K} (\rho(x, y, e)f)(k) = \sum_{k \in K} \chi(\tau_k^{(2)}(y, e))f(kx).$$

Concerning convergence of the series compare Proposition 5.1.

To generalize some results from abelian to non-abelian locally compact groups we need a replacement of the annihilator in the abelian case. Thus, for the remainder of this paper, let Γ_K be the subgroup of L defined by

$$\Gamma_K := \{m \in L : \chi(\tau_k^{(2)}(m, e)) = 1 \quad \text{for all } k \in K\}.$$

In the classical situation of a locally compact abelian group the subgroup Γ_K is just the annihilator of K in $\widehat{G}$ (compare Subsection 6.2).

4. Some basic facts

The next lemma is easily seen, but it is stated here explicitly, since it is used for all the following calculations.

Lemma 4.1. *Let* $\alpha : G \to \mathrm{Aut}(L \times Z)$ *be an action.*
(i) *The following conditions are equivalent.*
   (a) $Z(G \ltimes_\alpha (L \times Z)) = Z$.
   (b) *For all* $x \in G$ *and* $(y, z) \in L \times Z$, *we have* $\alpha_x(y, z) = (e, z)\alpha_x(y, e)$.
(ii) *Suppose that* $Z(G \ltimes_\alpha (L \times Z)) = Z$. *Then, for all* $x, x' \in G$, $y \in L$,

$$\alpha_x^{(2)}(\alpha_{x'}^{(1)}(y, e), e) = \alpha_{xx'}^{(2)}(y, e)(\alpha_{x'}^{(2)}(y, e))^{-1}.$$

It is desirable that the Zak transform of Definition 3.2 coincides with the classical Zak transform for $G = \mathbb{R}$. This is the subject of the following example.

Example 4.2. Let $G = \mathbb{R}$. Each non-trivial discrete subgroup of $\mathbb{R}$ is of the form $K = r\mathbb{Z}$ with $r \in \mathbb{R}^*$. Notice that $S_K := [0, r)$ is a relatively compact fundamental domain for K. Now let $L := Z := \mathbb{R}$. We intend to calculate possible Zak transforms in the sense of Definition 3.2.

Let $\tau : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ be an action. By Lemma 4.1 (i), there exist functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that τ is of the form

$$\tau_x = \begin{pmatrix} \varphi(x) & 0 \\ \psi(x) & 1 \end{pmatrix} \quad \text{for all } x \in \mathbb{R}.$$

Since τ is a homomorphism, we obtain, for all $x, y \in \mathbb{R}$,

$$(1) \qquad \varphi(x + y) = \varphi(x)\varphi(y)$$

and

$$(2) \qquad \psi(x + y) = \psi(x)\varphi(y) + \psi(y) = \psi(y)\varphi(x) + \psi(x).$$

Equation (1) and the fact that τ is an action imply that either $\varphi \equiv 0$ or there exists $a > 0$ such that, for all $x \in \mathbb{R}$,

$$\varphi(x) = a^x.$$

But $\tau_x \in \operatorname{Aut}(\mathbb{R}^2)$ for all $x \in \mathbb{R}$ implies $\varphi \not\equiv 0$. We have to consider two cases.

Case $a \neq 1$.

Then (2) can be rewritten in the following way

$$\psi(x)(a^y - 1) = \psi(y)(a^x - 1) \quad \text{for all } x, y \in \mathbb{R}.$$

Choosing $y = 1$ yields the existence of some $b \; (= \psi(1)) \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$\psi(x) = \frac{b}{a-1}(a^x - 1).$$

It is easily checked that (I) is satisfied and that (II) is fulfilled if and only if $b \neq 0$. Hence τ yields a Zak transform in the sense of Definition 3.2, which does not coincide with the classical Zak transform. But notice that the Zak transform associated with τ satisfies no quasi-periodicity relation, since the subgroup Γ_K is trivial.

Case $a = 1$.

Then (2) implies that ψ is a homomorphism and hence there exists $s \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$\psi(x) = sx.$$

Note that (I) is always fulfilled and that (II) is satisfied if and only if $s \neq 0$. Then $\mathbb{R} \ltimes_\tau (\mathbb{R} \times \mathbb{R})$ coincides with the Heisenberg group associated with $\mathbb{R}$. Hence ρ is the Schrödinger representation and therefore τ yields the classical Zak transform with the additional factor r. But this factor sometimes also appears in the definition of the Zak transform ([Jan88, Section 1]).

Hence we proved that, for $L = Z = \mathbb{R}$, Definition 3.2 does yield different Zak transforms, but the only ones fulfilling further important properties are the Zak transforms arising from the classical one with an additional factor. If L and Z are chosen in a different way, clearly this yields a different transform, for example, with $L = \mathbb{R}^2$ and $Z = \mathbb{R}$.

Moreover, the Zak transform on a locally compact abelian group is also a Zak transform in the sense of Definition 3.2. For this, compare Subsection 6.2.

Next we have to check whether the properties of the Zak transform in the abelian case carry over to the non-abelian case.

First, we will investigate whether the general Zak transform also satisfies a quasi-periodicity relation. For this, recall the definition of the subgroup Γ_K, which will replace the annihilator in the abelian case.

Proposition 4.3. *For all $f \in L^2(G)$ and $(x, y) \in G \times L, (l, m) \in K \times \Gamma_K$,*

$$Zf(lx, m\tau_l^{(1)}(y, e)) = \overline{\chi(\tau_l^{(2)}(y, e))} Zf(x, y).$$

Proof. Let $f \in L^2(G)$ and let $(x, y) \in G \times L, (l, m) \in K \times \Gamma_K$. The definition of Γ_K implies that

$$\sum_{k \in K} (\rho(l, m, e)f)(k) = \sum_{k \in K} \chi(\tau_k^{(2)}(m, e))f(kl) \overset{k \mapsto kl^{-1}}{=} \sum_{k \in K} f(k).$$

This yields

$$
\begin{aligned}
Zf(lx, m\tau_l^{(1)}(y, e)) &= \sum_{k \in K} (\rho((l, m, (\tau_l^{(2)}(y, e))^{-1})(x, y, e))f)(k) \\
&= \sum_{k \in K} \rho(l, m, (\tau_l^{(2)}(y, e))^{-1})(\rho(x, y, e)f)(k) \\
&= \overline{\chi(\tau_l^{(2)}(y, e))} \sum_{k \in K} \rho(l, m, e)(\rho(x, y, e)f)(k).
\end{aligned}
$$

Now the claim follows from the first part of the proof. $\square$

The next lemma establishes an important property of the connection between Γ_K and τ.

Lemma 4.4. *For all $k \in K$,*

$$
\tau_k^{(1)}(\Gamma_K \times \{e\}) = \Gamma_K.
$$

Proof. Let $k \in K$. First, we prove that $\tau_k^{(1)}(\Gamma_K \times \{e\}) < \Gamma_K$. For this, let $m \in \Gamma_K$. Then, by Lemma 4.1 (ii) and the definition of Γ_K,

$$
\chi(\tau_l^{(2)}(\tau_k^{(1)}(m, e), e)) = \overline{\chi(\tau_k^{(2)}(m, e))}\chi(\tau_{lk}^{(2)}(m, e)) = 1
$$

for all $l \in K$. This implies $\tau_k^{(1)}(m, e) \in \Gamma_K$.

Assume, towards a contradiction, that there exist $y \in L \backslash \Gamma_K$ and $k \in K$ such that $\tau_k^{(1)}(y, e) \in \Gamma_K$. This implies the existence of some $k_0 \in K$ such that

$$
\chi(\tau_{k_0}^{(2)}(y, e)) \neq 1 \text{ and also } 1 = \chi(\tau_l^{(2)}(\tau_k^{(1)}(y, e), e)) = \overline{\chi(\tau_k^{(2)}(y, e))}\chi(\tau_{lk}^{(2)}(y, e))
$$

for all $l \in K$. Hence

$$
\chi(\tau_{lk}^{(2)}(y, e)) = \chi(\tau_k^{(2)}(y, e))
$$

for all $l \in K$. Choosing $l \in K$ as $l := k_0 k^{-1}$, we obtain, by the choice of k_0,

$$
\chi(\tau_k^{(2)}(y, e)) \neq 1
$$

and hence, for all $l \in K$,

$$
\chi(\tau_{lk}^{(2)}(y, e)) \neq 1.
$$

With $l := k^{-1}$, we get a contradiction.

Summarizing, we proved $\tau_k^{(1)}(\Gamma_K \times \{e\}) < \Gamma_K$ and $\tau_k^{(1)}((L \backslash \Gamma_K) \times \{e\}) \subseteq L \backslash \Gamma_K$ for all $k \in K$. Applying (I) yields the claim. $\square$

Remark 4.5. Suppose that Γ_K is a uniform lattice in L. Then, by Lemma 2.1, there exists a relatively compact fundamental domain for Γ_K which we will denote by Ω_K. Hence every $y \in L$ can be uniquely written in the form $y = mt$ where $m \in \Gamma_K$ and $t \in \Omega_K$. Let $k \in K$. This implies, using (I) and Lemma 4.4, that each $y \in L$ can be uniquely written in the form $y = m_k t_k$ where $m_k \in \Gamma_K$ and $t_k \in \tau_k^{(1)}(\Omega_K \times \{e\})$. Thus, by Proposition 4.3, for all $f \in L^2(G)$, the map $Zf : G \times L \to \mathbb{C}$ is uniquely determined by its values on $S_K \times \Omega_K$.

5. Hilbert space isomorphism

In the abelian case it is important for many applications that the Zak transform $f \mapsto Zf|_{S_K \times \Omega_K}$ is an isometry. Under certain conditions the Zak transform is isometric also in the general situation.

Throughout this section suppose that G is unimodular and that Γ_K is a uniform lattice in L. By Lemma 2.1, there exists a relatively compact fundamental domain for Γ_K in L which we will denote by Ω_K. Let the Haar measure on G be normalized so that Weil's formula holds, when we take on G/K the normalized G-invariant Radon measure and the counting measure on K. Clearly, if G is σ-compact (equivalently, K is countable) then S_K has positive measure, ($|S_K| > 0$). However, this is also true in the general case. To see this, choose a compactly generated open subgroup H of G containing S_K and observe that $S_K k \cap H \neq \emptyset$ if and only if $k \in H$. Since H is σ-compact and K is discrete, there are only countably many $k \in K \cap H$. Thus H is contained in a countable union of sets $S_K k$, $k \in K$, whence $|S_K| > 0$.

The map $\Phi : S_K \to G/K$, $x \mapsto xK$, is a continuous bijection and, for each measurable subset M of S_K, Weil's formula gives

$$|M| = \int_G \chi_M(x)\,dx = \int_{G/K} \left(\sum_{k \in K} \chi_M(xk) \right) d(xK) = |\Phi(M)|.$$

This implies that Φ maps the measure on S_K induced by the Haar measure on G to the normalized measure on G/K.

In addition, let the Haar measure on L be normalized so that Weil's formula holds, if we take on L/Γ_K the normalized Haar measure and the counting measure on Γ_K. Similarly, we see that the induced measure on Ω_K is transformed into the Haar measure on L/Γ_K and $|\Omega_K| = 1$.

Proposition 5.1. *Retain the preceding assumptions and notations, and let $f \in L^2(G)$. Then, for almost all $(x, y) \in S_K \times \Omega_K$,*

$$Zf(x, \omega) = \sum_{k \in K} \chi(\tau_k^{(2)}(y, e))f(kx)$$

converges, and the function Zf belongs to $L^2(S_K \times \Omega_K)$ and satisfies $\|Zf\|_2 = \|f\|_2$.

Proof. For $k \in K$, define $f_k \in L^2(S_K \times \Omega_K)$ by

$$f_k(x, y) := \chi(\tau_k^{(2)}(y, e))f(kx).$$

Then we have

$$\sum_{k \in K} \|f_k\|_2^2 = \sum_{k \in K} \int_{S_K} \int_{\Omega_K} |f_k(x, y)|^2\,dy\,dx = \sum_{k \in K} \int_{S_K} |f(kx)|^2\,dx = \|f\|_2^2.$$

Now let $k, l \in K$ such that $k \neq l$. We claim that $\langle f_k, f_l \rangle = 0$. At first, we have

$$
\begin{aligned}
\langle f_k, f_l \rangle &= \int\limits_{S_K} \int\limits_{\Omega_K} f(kx)\overline{f(lx)} \chi(\tau_k^{(2)}(y, e)\tau_l^{(2)}(y^{-1}, e))\, dy\, dx \\
&= \int\limits_{S_K} f(kx)\overline{f(lx)}\, dx \int\limits_{\Omega_K} \chi(\tau_k^{(2)}(y, e)\tau_l^{(2)}(y^{-1}, e))\, dy.
\end{aligned}
$$

By [HR63, Lemma 23.19], for a compact abelian group C and for a non-trivial character φ of C, $\int_C \varphi(y) dy = 0$. Using the definition of Γ_K, we may apply this to $C = L/\Gamma_K$ and the character φ defined by

$$
\varphi(y\Gamma_K) = \chi(\tau_k^{(2)}(y, e)\tau_l^{(2)}(y^{-1}, e)), \quad y \in L.
$$

Note that φ is non-trivial by (II). We obtain

$$
\int\limits_{\Omega_K} \chi(\tau_k^{(2)}(y, e)\tau_l^{(2)}(y^{-1}, e))\, dy = \int_{L/\Gamma_K} \varphi(y\Gamma_K)\, d(y\Gamma_K) = 0,
$$

and this in turn implies $\langle f_k, f_l \rangle = 0$. It follows that the series $\sum_{k \in K} f_k$ converges in $L^2(S_K \times \Omega_K)$ and satisfies

$$
\| \sum_{k \in K} f_k \|_2^2 = \sum_{k \in K} \|f_k\|_2^2 = \|f\|_2^2.
$$

In particular, $Zf(x, \omega)$ exists for almost all $(x, \omega) \in S_K \times \Omega_K$. $\qquad\square$

As in the abelian case the Zak transform is even a Hilbert space isomorphism for a large class of locally compact groups. (In Section 6 it will turn out that (I) and (II) and the condition that Γ_K has to be a uniform lattice are not very restrictive.)

Theorem 5.2. *Retain the preceding assumptions and notations. If the set*

$$
\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}
$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$, then

$$
Z : L^2(G) \to L^2(S_K \times \Omega_K)
$$

is a Hilbert space isomorphism.

Proof. By Proposition 5.1, $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is an isometry. Obviously, Z is also linear. Hence, to prove that Z is a Hilbert space isomorphism, it remains to shows that $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ is also surjective.

Let $U \in \mathcal{U}(L^2(G))$ be defined by $Uf(t) = f(t^{-1})$. Now consider the set $M_1 \subseteq L^2(G)$, which is defined by

$$
\begin{aligned}
M_1 := \{\varphi_{l,m} &:= U^{-1}(\rho(l, m, e)(U\chi_{S_K})) \\
&= \chi(\tau_{-1}^{(2)}(m, e)) \cdot L_l \chi_{S_K} : (l, m) \in K \times \Gamma_K\}.
\end{aligned}
$$

Furthermore, consider the set $M_2 \subseteq L^2(S_K \times \Omega_K)$, which is defined by

$$
M_2 := \{Z\varphi_{l,m} : (l, m) \in K \times \Gamma_K\}.
$$

Now, for $f \in L^2(G)$ and $(x, y) \in G \times L$, $(l, m) \in K \times \Gamma_K$, we obtain

$$
\begin{aligned}
Z\varphi_{l,m}(x, y) &= \sum_{k \in K} \chi(\tau_k^{(2)}(y, e))\varphi_{l,m}(kx) \\
&= \sum_{k \in K} \chi(\tau_k^{(2)}(y, e))\chi(\tau_{(kx)^{-1}}^{(2)}(m, e))\chi_{S_K}(l^{-1}kx) \\
&\stackrel{k \mapsto lk}{=} \sum_{k \in K} \chi(\tau_{lk}^{(2)}(y, e))\chi(\tau_{(lkx)^{-1}}^{(2)}(m, e))\chi_{S_K}(kx).
\end{aligned}
$$

Since every $x \in G$ can be uniquely written in the form $x = ks$ where $k \in K$ and $s \in S_K$, $\chi_{S_K}(kx) \neq 0$ if and only if $k = e$. This implies, using Lemma 4.1 (ii) and the definition of Γ_K,

$$
\begin{aligned}
Z\varphi_{l,m}(x, y) &= \chi(\tau_l^{(2)}(y, e))\chi(\tau_{(lx)^{-1}}^{(2)}(m, e)) \\
&= \chi(\tau_l^{(2)}(y, e))\chi(\tau_{l^{-1}}^{(2)}(m, e))\chi(\tau_{x^{-1}}^{(2)}(\tau_{l^{-1}}^{(1)}(m, e), e)) \\
&= \chi(\tau_l^{(2)}(y, e))\chi(\tau_{x^{-1}}^{(2)}(\tau_{l^{-1}}^{(1)}(m, e), e)).
\end{aligned}
$$

By Lemma 4.4,

$$
(3) \qquad M_2 = \{\chi(\tau_l^{(2)}(y, e))\chi(\tau_{x^{-1}}^{(2)}(m, e)) : (l, m) \in K \times \Gamma_K\}.
$$

Since we supposed M_2 to be an orthonormal basis, the claim is proven. $\qquad \square$

Now we shall investigate when the set

$$
\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}
$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$. The next proposition gives a necessary and sufficient condition for this.

Proposition 5.3. *Retain the preceding assumptions and notations. Then the following conditions are equivalent.*

(i) *For all $m \in \Gamma_K$, $m \neq e$,*

$$
\int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m, e))\, dx = 0
$$

and the linear subspace

$$
\text{span}\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}
$$

is dense in $L^2(S_K \times \Omega_K)$.

(ii) *The set*

$$
\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}
$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

Proof. For simplicity, let $\phi_{l,m} \subseteq L^2(S_K \times \Omega_K)$ be defined by

$$
\phi_{l,m}(x, y) := \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e))
$$

for all $(x, y) \in S_K \times \Omega_K$ and $(l, m) \in K \times \Gamma_K$. Now, obviously, $\|\phi_{l,m}\|_2 = 1$ for all $(l, m) \in K \times \Gamma_K$. Next, for $(l_1, m_1), (l_2, m_2) \in K \times \Gamma_K$, we obtain

$$
\begin{aligned}
&\langle \phi_{l_1, m_1}, \phi_{l_2, m_2} \rangle \\
&= \int_{S_K} \int_{\Omega_K} \chi(\tau_{x^{-1}}^{(2)}(m_1, e) \tau_{l_1}^{(2)}(y, e)) \cdot \overline{\chi(\tau_{x^{-1}}^{(2)}(m_2, e) \tau_{l_2}^{(2)}(y, e))} \, dy \, dx \\
&= \int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m_1, e) \tau_{x^{-1}}^{(2)}(m_2^{-1}, e)) \, dx \cdot \int_{\Omega_K} \chi(\tau_{l_1}^{(2)}(y, e) \tau_{l_2}^{(2)}(y^{-1}, e)) \, dy.
\end{aligned}
$$

Using [HR63, Lemma 23.19] (compare the proof of Proposition 5.1), we obtain

$$
\int_{\Omega_K} \chi(\tau_{l_1}^{(2)}(y, e) \tau_{l_2}^{(2)}(y^{-1}, e)) \, dy = \delta_{l_1, l_2}.
$$

Moreover, for $l_1 = l_2$,

$$
\int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m_1, e) \tau_{x^{-1}}^{(2)}(m_2^{-1}, e)) \, dx = \int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m_1 m_2^{-1}, e)) \, dx
$$

This proves the equivalence of (i) and (ii). $\square$

6. Some special cases

Here we focus on some classes of examples of locally compact groups whose Zak transform is a Hilbert space isomorphism.

6.1. The case $\tau_x^{(1)}(\cdot, e) = \mathrm{Id}_L$ for all $x \in G$.

One important class of groups are those locally compact groups G, for which the associated action τ (compare (I)) satisfies $\tau_x^{(1)}(\cdot, e) = \mathrm{Id}_L$ for all $x \in G$. It will turn out that, if we want (II) to be satisfied, this condition forces G to be abelian. Moreover, we introduce conditions which are easier to check and which imply that the associated Zak transform is a Hilbert space isomorphism.

Now, for the remainder of this subsection, let G be a locally compact group and K a uniform lattice in G. Further, suppose that there exist locally compact abelian groups L and Z and some action $\tau : G \to \mathrm{Aut}(L \times Z)$ such that $Z(G \ltimes_\tau (L \times Z)) = Z$ and such that $\tau_x^{(1)}(\cdot, e) = \mathrm{Id}_L$ for all $x \in G$. Then, obviously, the map $y \mapsto \tau_k^{(1)}(y, e)$, $L \to L$, is an isomorphism for each $k \in K$. This implies (I). In addition, suppose that Γ_K is a uniform lattice in L. Let S_K and Ω_K be relatively compact fundamental domains for K in G and for Γ_K in L, respectively (compare Lemma 2.1).

The following basic lemma, which follows immediately from Lemma 4.1 (ii), will be used often throughout the next proofs.

Lemma 6.1. *For each $y \in L$, the map*

$$
x \mapsto \tau_x^{(2)}(y, e), \quad G \to Z,
$$

is a homomorphism.

The next proposition shows that in this case the action is constant on cosets of the commutator subgroup of G. Thus it is indeed an action of G modulo its commutator subgroup, hence of an abelian group.

Proposition 6.2. *Let $[G, G]$ denote the commutator subgroup of G. Then*

$$\tau : G/[G, G] \to \operatorname{Aut}(L \times Z).$$

Proof. Let $y \in L$ and let $h_y : G \to Z$ be defined by

$$h_y(x) = \tau_x^{(2)}(y, e).$$

By Lemma 6.1, h_y is a homomorphism. Since Z is abelian, we obtain $h_y(x) = e$ for all $x \in [G, G]$. Using Lemma 4.1 (i), this implies

$$\tau_x(y, z) = (y, \tau_x^{(2)}(y, z)) = (y, z) \quad \text{for all } x \in [G, G], (y, z) \in L \times Z.$$

This proves the claim. $\square$

Corollary 6.3. *Let $\chi \in \widehat{Z}$ be such that (II) holds. Then G is abelian.*

Proof. Towards a contradiction, assume that G is non-abelian. This implies that $[G, G]$, the commutator subgroup of G, is not trivial. By Proposition 6.2, $\chi(\tau_x^{(2)}(y, e)) = 1$ for all $x \in [G, G]$ and $y \in L$. This contradicts (II). $\square$

In the special situation discussed here it is easy to check whether condition (II) holds.

Proposition 6.4. *Let $\chi \in \widehat{Z}$. Then the following conditions are equivalent.*
  (i) *(II) holds.*
  (ii) *$S_{(1,\chi)} = \{e\}$.*

Proof. Note that, for $\chi \in \widehat{Z}$,

$$S_{(1,\chi)} = \{x \in G : \chi(\tau_x^{(2)}(y, e)) = 1 \text{ for all } y \in L\}.$$

Now the claim follows from Lemma 6.1. $\square$

Finally, we investigate when the set

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

Proposition 6.5. *Let $\chi \in \widehat{Z}$ be such that (II) holds. Then the following conditions are equivalent.*
  (i) *The set*

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

  is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

(ii) *For each $m \in \Gamma_K$, $m \neq e$, the character on G defined by $x \mapsto \chi(\tau_x^{(2)}(m, e))$ is non-trivial and we have*

$$\{(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)) : (l, m) \in K \times \Gamma_K\}$$

$$= A(K, \widehat{G}) \times A(\Gamma_K, \widehat{L}).$$

Proof. Let $m \in \Gamma_K$, $m \neq e$. Notice that, by Corollary 6.3, G is abelian. By Lemma 6.1, the character $x \mapsto \chi(\tau_x^{(2)}(m, e))$, $G \to \mathbb{T}$, is indeed a character on G/K. Hence, using [HR63, Lemma 23.19] (compare the proof of Proposition 5.1) and the normalization of the measure on S_K for Proposition 5.1, we obtain

$$\int_{S_K} \chi(\tau_{x^{-1}}^{(2)}(m, e)) \, dx = 0$$

if and only if the character mentioned above is non-trivial. This implies that the set in question is an orthonormal system in $L^2(S_K \times \Omega_K)$ if and only if, for each $m \in \Gamma_K$, $m \neq e$, the character $x \mapsto \chi(\tau_x^{(2)}(m, e))$ is non-trivial (compare Proposition 5.3 and its proof).

Now the map

$$(x, y) \mapsto \chi(\tau_{x^{-1}}^{(2)}(m, e))\chi(\tau_l^{(2)}(y, e)), \quad G \times L \to \mathbb{T},$$

is a character on $G \times L$ by Lemma 6.1 and belongs to $A(K, \widehat{G}) \times A(\Gamma_K, \widehat{L})$. $A(K, \widehat{G}) \times A(\Gamma_K, \widehat{L})$ is an orthonormal basis of $L^2(S_K \times \Omega_K)$. Applying Proposition 5.3 once more finishes the proof. $\square$

6.2. Locally compact abelian groups.

Let G be a locally compact abelian group which contains a uniform lattice K. We define L, Z and τ by $L := \widehat{G}$, $Z := \mathbb{T}$ and $\tau : G \to \mathrm{Aut}(\widehat{G} \times \mathbb{T})$, $\tau_x(\omega, z) := (\omega, z\omega(x))$. Let $\chi \in \widehat{\mathbb{T}} = \mathbb{Z}$ be defined by $\chi := 1$. Since the situation discussed here is a special case of the one studied in Subsection 6.1, it is easily checked that (I) and (II) are fulfilled. Hence the canonical Zak transform for a locally compact abelian group is a Zak transform in the sense of Definition 3.2. Furthermore, we obtain

$$\begin{aligned} \Gamma_K &= \{\gamma \in \widehat{G} : \chi(\tau_k^{(2)}(\gamma, e)) = 1 \quad \text{for all } k \in K\} \\ &= \{\gamma \in \widehat{G} : \gamma(k) = 1 \quad \text{for all } k \in K\}. \end{aligned}$$

This implies that in this case the set Γ_K is just the annihilator of K in $\widehat{G}$.

6.3. Connected and simply connected 2-step nilpotent Lie groups.

In this subsection we intend to define a Zak transform for all connected and simply connected 2-step nilpotent Lie groups. As a general reference to the theory of connected and simply connected 2-step nilpotent Lie groups and Lie algebras we mention [CG90] and [HN91].

Concerning the existence of uniform lattices in connected and simply connected nilpotent Lie groups, the following famous result of Malcev (see [Rag72, Theorem 2.12] and [Mal49]) provides a complete answer.

Theorem 6.6. *Let G be a simply connected nilpotent Lie group and let $\mathfrak{g}$ denote its Lie algebra. Then the following conditions are equivalent.*

(i) *G admits a uniform lattice.*

(ii) *$\mathfrak{g}$ admits a basis with respect to which the structure constants are rational.*

Hence let G be a connected and simply connected 2-step nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a basis with respect to which the structure constants are rational. This is not very restrictive as it is shown in [Nie83]. Furthermore, let K be a uniform lattice in G and let S_K be a relatively compact fundamental domain for K in G whose existence is guaranteed by Lemma 2.1.

In the following we present an example of locally compact abelian groups L and Z and some action τ such that the Zak transform associated with K, L, Z, τ and χ is a Hilbert space isomorphism.

First, recall that the Baker-Campbell-Hausdorff formula defines a group-multiplication $*$ on $\mathfrak{g}$. Since G is a simply connected nilpotent Lie group, G is isomorphic to $(\mathfrak{g}, *)$. Next let $\{X_1, \dots, X_n\}$ be a fixed basis of $\mathfrak{g}$ with respect to which the structure constants are rational and let $\{X_1^*, \dots, X_n^*\}$ be the dual basis of $\mathfrak{g}^*$. Now we may identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ with respect to this basis. Further, let $\exp : \mathfrak{g} \to G$ denote the exponential map.

For the remainder of this subsection, let $L = \mathbb{R}^n$ and $Z = \mathbb{R}$ and let $\chi \in \mathbb{R}^*$. Furthermore, let Ad^* denote the coadjoint representation of G on $\mathfrak{g}^*$. Recalling the preceding paragraph, we may define $\tau : G \to \mathrm{Aut}(L \times Z)$ by

$$\tau_x(y, z) = (\frac{1}{2}(y + \mathrm{Ad}_x^*(y)), z + \langle x, y \rangle).$$

Lemma 6.7. *In the above situation* (I) *and* (II) *hold.*

Proof. Note that, for proving (I), it remains to show that the map $y \mapsto \tau_k^{(1)}(y, e)$, $\mathbb{R}^n \to \mathbb{R}^n$, is an isomorphism for all $k \in K$. For all $X, Y \in \mathfrak{g}$ and $l \in \mathfrak{g}^*$, we obtain

$$((\mathrm{Ad}_{\exp X}^* l)(Y)) = l(\mathrm{Ad}_{(\exp X)^{-1}} Y) = l(-X * Y * X) = l(Y + [Y, X]).$$

This implies

$$\frac{1}{2}(l + \mathrm{Ad}_{\exp X}^* l)(Y) = l(Y + \frac{1}{2}[Y, X]).$$

Next, fix some $X \in \mathfrak{g}$ and define for $l \in \mathfrak{g}^*$, $\Phi_l, \Psi_l \in \mathfrak{g}^*$ by

$$\Phi_l(W) = l(W + \frac{1}{2}[W, X]) \quad \text{and} \quad \Psi_l(W) = l(W - \frac{1}{2}[W, X]).$$

It is easily checked that the map $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*, l \mapsto \Psi_l$, is the inverse of $\Phi : \mathfrak{g}^* \to \mathfrak{g}^*, l \mapsto \Phi_l$. Moreover, both maps are continuous. Thus Φ is an isomorphism. By definition of τ, this proves (I).

Moreover, notice that

$$x \mapsto e^{2\pi i \chi \langle x, \cdot \rangle}, \ G = (\mathfrak{g}, *) \to \widehat{\mathbb{R}^n},$$

is injective for all $\chi \in \mathbb{R}$, $\chi \neq 0$. Thus also (II) holds. $\square$

Next we consider the subgroup Γ_K of $\mathbb{R}^n$.

Lemma 6.8. Γ_K *is a uniform lattice in* $\mathbb{R}^n$.

Proof. First, $A(K, \widehat{\mathbb{R}^n})$ is discrete if and only if for each pair of elements $(u, v) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ there exist $\lambda \in \mathbb{R}$ and $k \in K$ such that $\langle u, k \rangle + \lambda \langle v, k \rangle \notin \mathbb{Z}$. This is equivalent to the fact that for each $v \in \mathbb{R}^n \setminus \{0\}$, there exists $k \in K$ such that $\langle v, k \rangle \neq 0$. Hence $A(K, \widehat{\mathbb{R}^n})$ is discrete if and only if K contains a basis of $\mathbb{R}^n$.

Secondly, $A(K, \widehat{\mathbb{R}^n})$ is cocompact if and only if there exists a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that $\langle v_i, k \rangle \in \mathbb{Z}$ for all $1 \leq i \leq n$ and $k \in K$. Now since $A(\{v_1, \ldots, v_n\}, \widehat{\mathbb{R}^n}) = A(\langle v_1, \ldots, v_n \rangle_{\mathbb{Z}}, \widehat{\mathbb{R}^n})$, we obtain the following. $A(K, \widehat{\mathbb{R}^n})$ is cocompact if and only if there exists a uniform lattice J in $\mathbb{R}^n$ such that $\langle x, k \rangle \in \mathbb{Z}$ for all $x \in J$ and $k \in K$. This in turn is equivalent to the fact that there exists a uniform lattice J in $\mathbb{R}^n$ such that $K \subseteq A(J, \widehat{\mathbb{R}^n})$.

The claim follows from Theorem 6.6. $\qquad\square$

Let Ω_K be a relatively compact fundamental domain for Γ_K in L (compare Lemma 2.1). It remains to check whether the set

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

Lemma 6.9. *The following conditions are equivalent.*

(i) *The set*

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$.

(ii) K *is a uniform lattice in* $(\mathfrak{g}, +)$.

Proof. Let $\tilde{K} := A(\Gamma_K, \widehat{\mathbb{R}^n})$. By Lemma 6.8, Γ_K is a uniform lattice in $\mathbb{R}^n$, hence also $\tilde{K}$ is a uniform lattice in $\mathbb{R}^n$. Further, let $S_{\tilde{K}}$ be some relatively compact fundamental domain for $\tilde{K}$. Notice that, by the proof of Lemma 6.8, we have $K \subseteq \tilde{K}$. Moreover, it is well-known that

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in \tilde{K} \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_{\tilde{K}} \times \Omega_K)$. Thus the set

$$\{(x, y) \mapsto \chi(\tau^{(2)}_{x^{-1}}(m, e))\chi(\tau^{(2)}_l(y, e)) : (l, m) \in K \times \Gamma_K\}$$

is an orthonormal basis of $L^2(S_K \times \Omega_K)$ if and only if $K = \tilde{K}$. This in turn is equivalent to (ii). $\qquad\square$

The previous results yield the following theorem.

Theorem 6.10. (i) *The Zak transform* $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ *associated with* K *is an isometry.*

(ii) *Suppose that* $K < (\mathfrak{g}, +)$. *Then the Zak transform* $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ *associated with* K *is a Hilbert space isomorphism.*

Proof. This follows immediately from Lemma 6.7, Lemma 6.8 and Lemma 6.9 applied to Proposition 5.1 and Theorem 5.2.   □

Now we investigate whether there exist uniform lattices in G which are subgroups of $(\mathfrak{g}, +)$. The next proposition shows that there indeed exist both uniform lattices which are subgroups of $(\mathfrak{g}, +)$ and which are not.

Proposition 6.11. *There exist uniform lattices K_1 and K_2 in G such that $K_1 \not< (\mathfrak{g}, +)$ and $K_2 < (\mathfrak{g}, +)$.*

Proof. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ with respect to which the structure constants are rational. Therefore, there exist $p_k^{ij} \in \mathbb{Z}$, $q_k^{ij} \in \mathbb{N}$, $1 \le i, j, k \le n$, such that

$$[X_i, X_j] = \sum_{k=1}^{n} \frac{p_k^{ij}}{q_k^{ij}} X_k \quad (1 \le i, j \le n).$$

Define $r_k \in \mathbb{Q}$, $1 \le k \le n$, by

$$r_k := \frac{\gcd\{p_k^{ij} : 1 \le i, j \le n\}}{\operatorname{lcm}\{q_k^{ij} : 1 \le i, j \le n\}}.$$

Further, let $Y_1, \ldots, Y_s \in \{\sum_{k=1}^{n} l_k r_k X_k : l_k \in \mathbb{Z}\}$, $1 \le s \le n$, be a basis of $\mathfrak{g}^1$, the commutator algebra of $\mathfrak{g}$, such that

$$\langle Y_1, \ldots, Y_s \rangle_{\mathbb{Z}} = \{\sum_{k=1}^{n} l_k r_k X_k : l_k \in \mathbb{Z}\} \cap \mathfrak{g}^1.$$

Without loss of generality we can assume that $\{X_1, \ldots, X_{n-s}, Y_1, \ldots, Y_s\}$ is a basis of $\mathfrak{g}$.

We define $K_1 \subseteq G$ by

$$K_1 := \{\sum_{i=1}^{n-s} m_i X_i + \sum_{j=1}^{s} l_j Y_j + \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n-s} m_i m_j [X_i, X_j] : m_i, l_j \in \mathbb{Z}$$

$$\text{for all } 1 \le i \le n - s, 1 \le j \le s\}.$$

First, we are going to prove that $K_1 < (\mathfrak{g}, *)$. Note that this implies $K_1 < G$. Let $m_i, m_i', l_j, l_j' \in \mathbb{Z}$ for all $1 \le i \le n - s, 1 \le j \le s$. Then

$$\left(\sum_{i=1}^{n-s} m_i X_i + \sum_{j=1}^{s} l_j Y_j + \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n-s} m_i m_j [X_i, X_j] \right)$$

$$* \left(\sum_{i=1}^{n-s} m_i' X_i + \sum_{j=1}^{s} l_j' Y_j + \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n-s} m_i' m_j' [X_i, X_j] \right)$$

$$= \sum_{i=1}^{n-s}(m_i + m_i')X_i + \sum_{j=1}^{s}(l_j + l_j')Y_j + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}(m_im_j + m_i'm_j')[X_i, X_j]$$

$$+ \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}(m_im_j' - m_i'm_j)[X_i, X_j]$$

$$= \sum_{i=1}^{n-s}(m_i + m_i')X_i + \sum_{j=1}^{s}(l_j + l_j')Y_j - \sum_{\substack{i,j=1 \\ i<j}}^{n-s}m_i'm_j[X_i, X_j]$$

$$+ \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}(m_i + m_i')(m_j + m_j')[X_i, X_j]$$

belongs to K_1, since

$$\sum_{\substack{i,j=1 \\ i<j}}^{n-s}m_i'm_j[X_i, X_j] = \sum_{k=1}^{n}\left[\sum_{\substack{i,j=1 \\ i<j}}^{n-s}m_i'm_j\frac{p_k^{ij}}{q_k^{ij}}\right]X_k$$

$$\in \{\sum_{k=1}^{n}l_kr_kX_k : l_k \in \mathbb{Z}\} \cap \mathfrak{g}^1 = \langle Y_1, \ldots, Y_s\rangle_{\mathbb{Z}}.$$

Obviously, this is even a uniform lattice in G.

To prove $K_1 \not< (\mathfrak{g}, +)$ assume, towards a contradiction, that $K_1 < (\mathfrak{g}, +)$. Let $u, v \in \{1, \ldots, n\}$ be arbitrary but fixed. This implies that $X_u + X_v$ belongs to K_1, since $X_u, X_v \in K_1$. Hence there exist $m_i, l_j \in \mathbb{Z}$ for all $1 \leq i \leq n-s, 1 \leq j \leq s$, such that

$$X_u + X_v = \sum_{i=1}^{n-s}m_iX_i + \sum_{j=1}^{s}l_jY_j + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}m_im_j[X_i, X_j].$$

Since $\{X_1, \ldots, X_{n-s}, Y_1, \ldots, Y_s\}$ is a basis of $\mathfrak{g}$ and $\{Y_1, \ldots, Y_s, [X_i, X_j] : 1 \leq i, j \leq n\} \subseteq \mathfrak{g}^1$, we obtain

$$m_u = m_v = 1 \text{ and } m_i = 0 \text{ for all } 1 \leq i \leq n - s, i \neq u, v.$$

Thus there exist $l_j \in \mathbb{Z}$ for all $1 \leq j \leq s$, such that

$$\sum_{j=1}^{s}l_jY_j + \frac{1}{2}[X_u, X_v] = 0.$$

But, by the choice of r_k, there exist $u, v \in \{1, \ldots, n\}$ such that

$$\sum_{k=1}^{n}\frac{1}{2}\frac{p_k^{uv}}{q_k^{uv}}X_k \notin \langle Y_1, \ldots, Y_s\rangle_{\mathbb{Z}},$$

a contradiction.

Now we turn to the set $K_2 \subseteq G$ defined by

$$K_2 := \{\sum_{i=1}^{n-s} m_i X_i + \frac{1}{2} \sum_{j=1}^{s} l_j Y_j : m_i, l_j \in \mathbb{Z} \text{ for all } 1 \le i \le n-s, 1 \le j \le s\}.$$

For $m_i, m_i', l_j, l_j' \in \mathbb{Z}$, $1 \le i \le n-s$, $1 \le j \le s$,

$$\left(\sum_{i=1}^{n-s} m_i X_i + \frac{1}{2} \sum_{j=1}^{s} l_j Y_j\right) * \left(\sum_{i=1}^{n-s} m_i' X_i + \frac{1}{2} \sum_{j=1}^{s} l_j' Y_j\right)$$

$$= \sum_{i=1}^{n-s}(m_i + m_i')X_i + \frac{1}{2}\sum_{j=1}^{s}(l_j + l_j')Y_j + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{n-s}(m_i m_j' - m_i' m_j)[X_i, X_j]$$

belongs to K_2 by the same argument as above. This implies $K_2 < G$. Moreover, K_2 is also a uniform lattice in G. Notice that K_2 coincides with the lattice in $\mathfrak{g}$ generated by $\{X_1, \ldots, X_{n-s}, \frac{1}{2}Y_1, \ldots, \frac{1}{2}Y_s\}$. Thus $K_2 < (\mathfrak{g}, +)$. $\square$

Remark 6.12. Let $G = \mathbb{R}$ and $K = \mathbb{Z}$. Then the action τ defined here yields the classical Zak transform.

The action τ seems to be defined arbitrarily, but its definition is quite natural as we will see in a moment. Recall that the classical Heisenberg group can be constructed using the so-called position and momentum operator. These operators together with the identity operator generate a Lie algebra and then the classical Heisenberg group is defined to be the associated Lie group [Fol89]. This construction may be applied to a more general setting, namely when $\mathbb{R}$ is replaced by an arbitrary connected and simply connected 2-step nilpotent Lie group G. This was done by Folland [Fol94, Section 2]. In the following paragraph we state the main steps of the construction, since we will use them thereafter.

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$. Moreover, let $c_k^{ij} \in \mathbb{R}$, $1 \le i, j, k \le n$, be defined by

$$[X_i, X_j] = \sum_{k=1}^{n} c_k^{ij} X_k \quad \text{for all } 1 \le i, j \le n.$$

As mentioned in the beginning of this subsection we may identify G with $(\mathfrak{g}, *)$. Let $\mathcal{S}(G)$ denote the class of Schwartz functions on G. The operators $Q_1, \ldots, Q_n : \mathcal{S}(G) \to \mathcal{S}(G)$ defined by

$$Q_i f(x_1, \ldots, x_n) = x_i f(x_1, \ldots, x_n), \ i = 1, \ldots, n,$$

will replace the position operator. Moreover, the operators $P_1, \ldots, P_n : \mathcal{S}(G) \to \mathcal{S}(G)$ defined by

$$P_i f(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{j=1}^{n} c_k^{ji} x_j\right)\frac{\partial f}{\partial x_k}(x_1, \ldots, x_n)$$

for all $i \in \{1, \ldots, n\}$ will play the role of the momentum operator. These operators are derivatives, which are left-invariant, that means

$$P_i(L_x f) = L_x(P_i f) \quad (x \in G, 1 \leq i \leq n).$$

Further, let $I : \mathcal{S}(G) \to \mathcal{S}(G)$ be defined by

$$I = \mathrm{Id}_{\mathcal{S}(G)}.$$

The operators $\{Q_i, P_i, I : 1 \leq i \leq n\}$ generate a 3-step nilpotent Lie algebra $\mathfrak{h}$ with

$$[P_i, P_j] = \sum_{k=1}^{n} c_k^{ij} P_k \quad \text{for all } 1 \leq i, j \leq n,$$

$$[P_i, Q_i] = I + \frac{1}{2} \sum_{k=1}^{n} c_i^{ki} Q_k \quad \text{for all } 1 \leq i \leq n \quad \text{and}$$

$$[P_i, Q_j] = \frac{1}{2} \sum_{k=1}^{n} c_j^{ki} Q_k \quad \text{for all } 1 \leq i, j \leq n, i \neq j.$$

By the Baker-Campbell-Hausdorff formula, the Lie group associated with $\mathfrak{h}$ is isomorphic to $\mathfrak{h}$ as a set endowed with the following group multiplication

$$\left(\sum_{k=1}^{n} x_k P_k + \sum_{k=1}^{n} y_k Q_k + zI \right) * \left(\sum_{k=1}^{n} x_k' P_k + \sum_{k=1}^{n} y_k' Q_k + z'I \right)$$

$$= \sum_{k=1}^{n} (x_k + x_k' + \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j' c_k^{ij}) P_k + \sum_{k=1}^{n} (y_k + y_k' + \frac{1}{4} \sum_{i,j=1}^{n} (x_i y_j' - y_j x_i') c_j^{ki}) Q_k$$

$$+ (z + z' + \frac{1}{2} \sum_{i=1}^{n} (x_i y_i' - y_i x_i') + \frac{1}{8} \sum_{i,j,k=1}^{n} (y_k' - y_k) x_i x_j' c_k^{ij}) I.$$

From the preceding paragraph we see that $\mathfrak{g} = \langle P_1, \ldots, P_n \rangle$ is a subalgebra of $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is even a semidirect product, where $\mathfrak{g}$ acts on $\langle Q_1, \ldots, Q_n, I \rangle$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{R}$ and define $X, Y \in \mathfrak{h}$ by $X = x_1 P_1 + \ldots + x_n P_n$, $Y = y_1 Q_1 + \ldots + y_n Q_n + zI$. Then the action τ of $\mathfrak{g}$ on $\langle Q_1, \ldots, Q_n, I \rangle$ is given by

$$\tau_X(Y) = X * Y * (-X)$$

$$= \sum_{k=1}^{n} (y_k + \frac{1}{2} \sum_{i,j=1}^{n} x_i y_j c_j^{ki}) Q_k + (z + \langle x, y \rangle) I.$$

We claim that

$$\frac{1}{2}(y + \mathrm{Ad}_x^*(y)) = (y_k + \frac{1}{2} \sum_{i,j=1}^{n} x_i y_j c_j^{ki})_{1 \leq k \leq n}$$

for all $x \in G$, $y \in \mathbb{R}^n$.

For this, let $X = x_1 P_1 + \ldots + x_n P_n, W = w_1 P_1 + \ldots + w_n P_n \in \mathfrak{g}$ and let $l = y_1 P_1^* + \ldots + y_n P_n^* \in \mathfrak{g}^*$. Then, by using the proof of Lemma 6.7,

$$
\begin{aligned}
\frac{1}{2}(l + \mathrm{Ad}^*_{\exp X} l)(W) &= l(W + \frac{1}{2}[W, X]) \\
&= l(\sum_{k=1}^{n}(w_k + \frac{1}{2}\sum_{i,j=1}^{n} c_k^{ij} w_i x_j) P_k) \\
&= \sum_{k=1}^{n}(w_k y_k + \frac{1}{2}\sum_{i,j=1}^{n} c_k^{ij} y_k w_i x_j) \\
&= \sum_{k=1}^{n}(y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki}) w_k \\
&= \tilde{l}(W),
\end{aligned}
$$

where

$$
\tilde{l} = \sum_{k=1}^{n}(y_k + \frac{1}{2}\sum_{i,j=1}^{n} x_i y_j c_j^{ki}) P_k^*.
$$

ACKNOWLEDGMENT

REFERENCES

[AGT92] L. Auslander, I. Gertner and R. Tolimieri, *The finite Zak transform and the finite Fourier transform*, Radar and sonar, Pt. II, Proc. Lect. IMA Summer Progr., Minneapolis/MN (USA) 1990, IMA Vol. Math. Appl. **39** (1992), 21-35.

[CG90] L. Corwin and F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples*, Cambridge University Press, Cambridge, 1990.

[FS98] H.G. Feichtinger and T. Strohmer, *Gabor Analysis and Algorithms, Theory and Applications*, Birkhäuser, Boston/Basel/Berlin, 1998.

[Fol89] G.B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, 1989.

[Fol94] G.B. Folland, *Meta-Heisenberg groups*, In *Fourier analysis: analytic and geometric aspects*, edited by W.O. Bray, P.S. Milojevic and C.V. Stanojevic, Lect. Notes Pure Appl. Math. **157**, Dekker, New York (1994), 121-147.

[Fol95] G.B. Folland, *A course in abstract harmonic analysis*, CRC Press, Boca Raton, 1995.

[Gel50] I.M. Gelfand, *Eigenfunction expansions for equations with periodic coefficients*, Dokl. Akad. Nauk. SSR **73** (1950), 1117-1120.

[Hei89] C. Heil, *A discrete Zak transform*, Technical Report MTR-89W00281, The MITRE Corporation, 1989.

[HR63] E. Hewitt and K.A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin/Heidelberg/New York, 1963.

[HN91] J. Hilgert and K.-H. Neeb, *Lie-Gruppen und Lie-Algebren*, Vieweg, Braunschweig/Wiesbaden, 1991.

[Jan88] A.J.E.M. Janssen, *The Zak transform: A signal transform for sampled time-continuous signals*, Philips J. Res. **43** (1988), 23-69.

[KK98] E. Kaniuth and G. Kutyniok, *Zeros of the Zak transform on locally compact abelian groups*, Proc. Amer. Math. Soc. **126** (1998), 3561-3569.

[Ku00] G. Kutyniok, *Time-frequency analysis on locally compact groups*, Ph.D. thesis, University of Paderborn, 2000.

[Mal49] A.I. Malcev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat. **13** (1949), 9-32.

[Nie83] O.A. Nielsen, *Unitary representations and coadjoint orbits of low-dimensional nilpotent Lie groups*, In Queen's Papers in Pure and Appl. Math., Vol. 63, Queen's Univ., Kingston, ON, 1983.

[Rag72] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, Berlin/Heidelberg/New York, 1972.

[Wei64] A. Weil, *Sur certains groups d'opérateurs unitaires*, Acta Math. **111** (1964), 143-211.

[Zak67] J. Zak, *Finite translations in solid state physics*, Phys. Rev. Letters **19** (1967), 1385-1387.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF PADERBORN, 33095 PADERBORN, GERMANY

E-mail address: gittak@uni-paderborn.de