LINEAR INDEPENDENCE OF TIME-FREQUENCY SHIFTS UNDER A GENERALIZED SCHRÖDINGER REPRESENTATION

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Abstract. Let $\rho_{\mathbb{R}}$ be the classical Schrödinger representation of the Heisenberg group and let Λ be a finite subset of $\mathbb{R} \times \mathbb{R}$. The question of when the set of functions $\{t \mapsto e^{2\pi i y t} f(t+x) = (\rho_{\mathbb{R}}(x,y,1)f)(t) : (x,y) \in \Lambda\}$ is linearly independent for all $f \in L^2(\mathbb{R})$, $f \neq 0$, arises from Gabor analysis. We investigate an analogous problem for locally compact abelian groups G. For a finite subset Λ of $G \times \widehat{G}$ and ρ_G the Schrödinger representation of the Heisenberg group associated with G, we give a necessary and in many situations also sufficient condition for the set $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ to be linearly independent for all $f \in L^2(G)$, $f \neq 0$.

1. Introduction

An important problem in Gabor analysis is the implementation of frames [1]. In fact, any practical implementation has to be finite. Since any finite collection of linearly independent vectors is a Riesz basis for its linear span, the following question arises. Let Λ be a finite subset of $\mathbb{R} \times \mathbb{R}$ and let $f \in L^2(\mathbb{R})$, $f \neq 0$. When is the set of functions

$$t \mapsto e^{2\pi i y t} f(t+x), \quad (x,y) \in \Lambda,$$

in $L^2(\mathbb{R})$ linearly independent?

This problem has been investigated by Heil, Ramanathan and Topiwala in [5]. In particular, they proved that such a set is linearly independent, if Λ is a finite subset of a unit lattice in $\mathbb{R} \times \mathbb{R}$. This case is especially important for questions concerning frames. Later Linnell [9] extended this result to finite subsets Λ of discrete subgroups of $\mathbb{R} \times \mathbb{R}$.

In [5] it was pointed out that this problem should be studied for more general locally compact abelian groups, since often frames in Hilbert spaces other than $L^2(\mathbb{R})$ have to be considered. Now, this problem admits a natural generalization to these groups.

Let G be a locally compact abelian group, let $\widehat{G}$ denote its dual group and let ρ_G denote the Schrödinger representation associated with G. For which finite subsets Λ of $G \times \widehat{G}$ and for which functions $f \in L^2(G)$, $f \neq 0$, is the subset $\{\rho_G(x,\omega,1)f : (x,\omega) \in \Lambda\}$ of $L^2(G)$ linearly independent?

Heil, Ramanathan and Topiwala conjectured in [5], that $\{\rho_{\mathbb{R}}(x,y,1)f : (x,y) \in \Lambda\}$ is linearly independent for all finite subsets Λ of $\mathbb{R}\times\mathbb{R}$ and for all functions $f \in L^2(\mathbb{R})$,

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$f \neq 0$. It turns out (see Theorems 1 and 2) that the appropriate conjecture for locally compact abelian groups should be the following one.

Conjecture. *Let G be a locally compact abelian group, let Λ be a finite subset of $G \times \widehat{G}$ and let G^c and $(\widehat{G})^c$ denote the set of compact elements in G and $\widehat{G}$, respectively. Then the following conditions are equivalent.*

(I) *The subset $\{\rho_G(x, \omega, 1)f : (x, \omega) \in \Lambda\}$ of $L^2(G)$ is linearly independent for all $f \in L^2(G)$, $f \neq 0$.*

(II) *The elements $(x, \omega)(G^c \times (\widehat{G})^c)$, $(x, \omega) \in \Lambda$, are pairwise different.*

The first theorem proves, in particular, the implication (I) $\Rightarrow$ (II) of the conjecture.

Theorem 1. *Let G be a locally compact abelian group and let Λ be a finite subset of $G \times \widehat{G}$. Suppose that, for each $f \in C_c(G)$, $f \neq 0$, the subset*

$$\{\rho_G(x, \omega, 1)f : (x, \omega) \in \Lambda\}$$

of $L^2(G)$ is linearly independent. Then (II) *holds.*

We did not succeed to prove the converse direction. This remains even open for $G = \mathbb{R}$ (compare [5]). But we are going to establish the sufficiency of (I) in the following case, which is especially important for applications.

Theorem 2. *Let G be a locally compact abelian group, let K be a uniform lattice in G and let $A(K, \widehat{G})$ denote the annihilator of K in $\widehat{G}$. Furthermore, let Λ be a finite subset of $K \times A(K, \widehat{G})$. Then* (I) *is equivalent to* (II).

The proofs of Theorem 1 and Theorem 2 are given in Sections 3 and 4, respectively. Both theorems will be proven by first reducing to locally compact abelian Lie groups and then to compactly generated locally compact abelian Lie groups. Note that the set of compactly generated locally compact abelian Lie groups coincides with the set of elementary LCA groups dealt with in [3, Chapter 7]. Finally, in Section 5 we conclude with some remarks.

2. Preliminaries

Let G be a locally compact group. If $G = H_1 \times \ldots \times H_n$, $n \in \mathbb{N}$, then let $x_i \in H_i$ denote the i^{th} component of $x \in G$ for all $1 \leq i \leq n$. Let $C(G)$ denote the space of continuous functions on G, $C_c(G)$ the space of continuous functions with compact support on G and $L^2(G)$ the space of square integrable functions on G.

An element $x \in G$ is said to be *compact*, if the smallest closed subgroup of G containing x is compact. Let G^c denote the set of compact elements in G. When G is abelian, G^c is always a closed subgroup [6, Theorem 9.10].

Remark 2.1. Let G be a locally compact group, K a compact normal subgroup of G and $q : G \to G/K$ the quotient homomorphism. Then $G^c = q^{-1}((G/K)^c)$, and hence for any two elements $x, y \in G$, $xG^c \neq yG^c$ if and only if $q(x)(G/K)^c \neq q(y)(G/K)^c$.

Throughout the paper let G be a locally compact abelian group with dual group denoted by $\widehat{G}$. As a general reference to duality theory of locally compact abelian groups we mention [6].

A subgroup K of G will be called a *uniform lattice*, if it is discrete and cocompact. The subgroup $A(K, \widehat{G}) = \{\omega \in \widehat{G} : \omega(k) = 1 \text{ for all } k \in K\}$ is called the *annihilator* *of* K *in* $\widehat{G}$. Let K be a uniform lattice in G. Then, since $A(K, \widehat{G}) = \widehat{G/K}$ and $\widehat{G}/A(K, \widehat{G}) = \widehat{K}$ and since the dual of a compact abelian group is discrete and vice versa, the subgroup $A(K, \widehat{G})$ is a uniform lattice in $\widehat{G}$.

The *Heisenberg group associated with* G, $H(G)$, is the set $G \times \widehat{G} \times \mathbb{T}$ with multiplication defined by

$$(x, \omega, z)(x', \omega', z') = (xx', \omega\omega', zz'\omega'(x)).$$

In the following we will consider the so-called *Schrödinger representation*, which is the irreducible unitary representation of $H(G)$ on $L^2(G)$ defined by

$$(\rho_G(x, \omega, z)f)(t) = z\,\omega(t) f(xt).$$

This is the natural generalization of the Schrödinger representation of $H(\mathbb{R})$ dealt with in Gabor analysis.

3. Proof of Theorem 1

We shall begin with the special case of a compactly generated locally compact abelian Lie group.

Proposition 3.1. *Let G be a compactly generated locally compact abelian Lie group. Then the conclusion of Theorem 1 holds.*

Proof. Suppose that the functions $\rho_G(x, \omega, 1)f$, $(x, \omega) \in \Lambda$, are linearly independent for every $f \in C_c(G)$, $f \neq 0$. Towards a contradiction, assume that there are two distinct elements $(x, \omega) \in \Lambda$ and $(x', \omega') \in \Lambda$ such that

$$(x, \omega)(G^c \times (\widehat{G})^c) = (x', \omega')(G^c \times (\widehat{G})^c).$$

The structure theorem for compactly generated locally compact abelian Lie groups implies that G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where F is a finite group and $p, q, r \in \mathbb{N}_0$. Hence the assumption says that $x_1 = x_1'$, $x_2 = x_2'$, $\omega_1 = \omega_1'$ and $\omega_3 = \omega_3'$.

Now we have to construct a function $f \in C_c(G)$, $f \neq 0$, such that $\rho_G(x, \omega, 1)f$ and $\rho_G(x', \omega', 1)f$ are linearly dependent. For that, fix $f_1 \in C_c(\mathbb{R}^p)$, $f_1 \neq 0$, and let $f_4 : F \to \mathbb{C}$, $f_4 \neq 0$, be arbitrary. f_4 will be specified later. Then consider a function $f \in C_c(G)$, $f \neq 0$, of the form

$$f(t) \;:=\; \begin{cases} f_1(t_1) f_4(t_4) & : \quad t_2 = 0, \\ 0 & : \quad t_2 \neq 0 \end{cases}$$

for all $t \in G$. Let $\lambda \in \mathbb{C}^2$, $\lambda \neq 0$. Using the definition of the representation ρ_G and the assumption that $x_1 = x_1'$, $x_2 = x_2'$, $\omega_1 = \omega_1'$ and $\omega_3 = \omega_3'$, we obtain that

$$\lambda_1(\rho_G(x, \omega, 1)f)(t) + \lambda_2(\rho_G(x', \omega', 1)f)(t) = 0 \quad \text{for almost all } t \in G$$

is equivalent to

$$\begin{aligned} &\lambda_1 \omega_2(t_2)\omega_4(t_4) f(x_1 + t_1, x_2 + t_2, x_3 t_3, x_4 t_4) \\ &\quad + \lambda_2 \omega_2'(t_2)\omega_4'(t_4) f(x_1 + t_1, x_2 + t_2, x_3' t_3, x_4' t_4) = 0 \quad \text{for all } t \in G. \end{aligned} \tag{1}$$

By the construction of f and the fact that $f_1 \neq 0$, (1) holds if and only if

$$\lambda_1 \overline{\omega_2(x_2)}\omega_4(t_4) f_4(x_4 t_4) + \lambda_2 \overline{\omega_2'(x_2)}\omega_4'(t_4) f_4(x_4' t_4) = 0 \quad \text{for all } t_4 \in F. \tag{2}$$

In order to simplify this equation, replace t_4 by ${x'_4}^{-1}t_4$ and define $\tilde{\lambda} \in \mathbb{C}^2$ by $\tilde{\lambda}_1 := \lambda_1\overline{\omega_2(x_2)}\,\overline{\omega_4(x'_4)}$ and $\tilde{\lambda}_2 := \lambda_2\overline{\omega'_2(x_2)}\,\overline{\omega'_4(x'_4)}$. Then (2) is equivalent to

$$\tilde{\lambda}_1\omega_4(t_4)f_4((x_4{x'_4}^{-1})t_4) + \tilde{\lambda}_2\omega'_4(t_4)f_4(t_4) = 0 \quad \text{for all } t_4 \in F.$$

Thus, towards a contradiction, it remains to show that for arbitrary $x \in F$ and $\omega \in \widehat{F}$ there exist some $\mu \in \mathbb{C}$, $\mu \neq 0$, and a function $g : F \to \mathbb{C}$, $g \neq 0$, such that

$$g(xt) = \mu\,\omega(t)g(t) \quad \text{for all } t \in F. \tag{3}$$

Since F is finite, there exists a minimal $N \in \mathbb{N}$ such that $x^N = e$ and then $H := \{x^n : 0 \leq n \leq N-1\}$ is a subgroup of F. Define $\mu \in \mathbb{C}$ such that

$$\mu^N\omega(x)^{\frac{N(N-1)}{2}} = 1,$$

and define $g : F \to \mathbb{C}$ by

$$g(t) := \begin{cases} \mu^n\omega(x)^{\frac{n(n-1)}{2}} & : \quad t = x^n \in H,\ 0 \leq n \leq N-1, \\ 0 & : \quad t \notin H. \end{cases}$$

Then (3) holds for all $t \in F\backslash H$, because then $xt \in F\backslash H$. If, on the other hand, $t = x^n \in H$, we have that

$$g(x^{n+1}) = \mu^{n+1}\omega(x)^{\frac{n(n+1)}{2}} = \mu\,\omega(x^n)g(x^n)$$

in the case $0 \leq n < N-1$. And in the case $n = N-1$ we obtain

$$g(x^N) = g(x^0) = 1 = \mu^N\omega(x)^{\frac{N(N-1)}{2}} = \mu\,\omega(x^{N-1})g(x^{N-1}).$$

Thus equation (3) is fulfilled, which finishes the proof. □

Now we can complete the proof of Theorem 1 using Proposition 3.1. Let G be a locally compact abelian group. Suppose that there are two distinct elements (x, ω) and (x', ω') of Λ such that

$$(x, \omega)(G^c \times (\widehat{G})^c) = (x', \omega')(G^c \times (\widehat{G})^c).$$

We intend to construct a suitable function $f \in C_c(G)$, $f \neq 0$, in such a way that $\rho_G(x, \omega, 1)f$ and $\rho_G(x', \omega', 1)f$ are linearly dependent.

This will be achieved by first reducing to locally compact abelian Lie groups and then to compactly generated locally compact abelian Lie groups, in which situation Proposition 3.1 applies.

Since there exists a compact subgroup K of G such that G/K is a Lie group and since each compact abelian group is a projective limit of Lie groups (compare [6, 28.61 (c)]), also G is a projective limit of Lie groups. Therefore there exists a system $\mathcal{H}$ of compact subgroups H of G, $\mathcal{H}$ downwards directed and $\bigcap_{H\in\mathcal{H}} H = \{e\}$, such that G/H is a Lie group for every $H \in \mathcal{H}$. Since $\widehat{G} = \bigcup_{H\in\mathcal{H}} \widehat{G/H}$, there exists $H \in \mathcal{H}$ such that $\omega, \omega' \in \widehat{G/H}$. Let $\pi : G \to G/H$ denote the quotient map. Then, by Remark 2.1, $xx'^{-1} \in G^c = \pi^{-1}((G/H)^c)$, hence $\pi(x)\pi(x')^{-1} \in (G/H)^c$. Similarly, $\omega\overline{\omega'} \in (\widehat{G})^c \cap \widehat{G/H} = (\widehat{G/H})^c$. So we obtain that

$$(\pi(x), \omega)\left((G/H)^c \times (\widehat{G/H})^c\right) = (\pi(x'), \omega')\left((G/H)^c \times (\widehat{G/H})^c\right).$$

Suppose that we have found $g \in C_c(G/H)$, $g \neq 0$, such that $\rho_{G/H}(\pi(x), \omega, 1)g$ and $\rho_{G/H}(\pi(x'), \omega', 1)g$ are linearly dependent. Let $f = g \circ \pi$. Then $f \in C_c(G)$, $f \neq 0$, and

$$(\rho_{G/H}(\pi(y), \chi, 1)g) \circ \pi = \rho_G(y, \chi, 1)f$$

for all $(y, \chi) \in G \times \widehat{G/H}$. Hence we can assume that G is a Lie group.

For the next step let L be an open, compactly generated subgroup of G, with the property that x, $x' \in L$. Then $xx'^{-1} \in G^c$ implies that $xx'^{-1} \in L^c$. We also have that $\omega|_L\overline{\omega'|_L} \in (\widehat{L})^c$, because the restriction map $\widehat{G} \to \widehat{L}$ is continuous. Hence

$$(x, \omega|_L)(L^c \times (\widehat{L})^c) = (x', \omega'|_L)(L^c \times (\widehat{L})^c).$$

Now we may apply Proposition 3.1 to this situation. This yields a function $g \in C_c(L)$, $g \neq 0$, and $\lambda \in \mathbb{C}^2$, $\lambda \neq 0$, such that

$$\lambda_1 \rho_L(x, \omega|_L, 1)g + \lambda_2 \rho_L(x', \omega'|_L, 1)g = 0. \tag{4}$$

Let $f \in C_c(G)$ be the function which equals g on L and is zero on $G \backslash L$. Then $\text{supp} f \subseteq L$. Since x and x' are elements in L, for every $t \in G$, we have that $xt, x't \in L$ if and only if $t \in L$. So it follows from (4) that

$$\lambda_1 \rho_G(x, \omega, 1)f + \lambda_2 \rho_G(x', \omega', 1)f = 0,$$

a contradiction.

4. Proof of Theorem 2

As before, we first consider compactly generated locally compact abelian Lie groups.

Proposition 4.1. *Let G be a compactly generated locally compact abelian Lie group. Then the conclusion of Theorem 2 holds.*

Proof. The implication (I) $\Rightarrow$ (II) was already proven in Proposition 3.1. Thus it remains to show that (II) implies (I).

For this let Λ be a finite subset of $K \times A(K, \widehat{G})$. Let $f \in L^2(G)$, $f \neq 0$, and $(\lambda_{(x,\omega)})_{(x,\omega)\in\Lambda} \subseteq \mathbb{C}$ be such that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}(\rho_G(x, \omega, 1)f) = 0$$

on G. Suppose that the elements $(x, \omega)(G^c \times (\widehat{G})^c)$, $(x, \omega) \in \Lambda$, are pairwise different. We have to prove that $\lambda_{(x,\omega)} = 0$ for all $(x, \omega) \in \Lambda$.

By [7, Lemma 2], there exists a relatively compact *fundamental domain* S_K of K in G which means that S_K is a relatively compact Borel subset of G such that every $y \in G$ can be uniquely written in the form $y = sk$ where $s \in S_K$ and $k \in K$. By Section 2, the subgroup $A(K, \widehat{G})$ is a uniform lattice in $\widehat{G}$, hence there similarly exists a relatively compact fundamental domain Ω_K of $A(K, \widehat{G})$ in $\widehat{G}$. The map $Z : L^2(G) \to L^2(S_K \times \Omega_K)$ defined by

$$Zf(y, \chi) = \sum_{k \in K} f(yk)\chi(k), \quad (y, \chi) \in S_K \times \Omega_K,$$

is called the *Zak transform associated with* K ([3, Definition 6.4.1], [7, p.3566]). Since this transform is isometric (see [7, Lemma 3]), we have that

$$Z\left(\sum_{(x,\omega)\in\Lambda}\lambda_{(x,\omega)}(\rho_G(x,\omega,1)f)\right)(y,\chi)=0 \text{ for almost all } (y,\chi)\in S_K\times\Omega_K.$$

Hence, by a simple calculation,

$$\left(\sum_{(x,\omega)\in\Lambda}\lambda_{(x,\omega)}\omega(y)\overline{\chi(x)}\right)Zf(y,\chi)=0 \text{ for almost all } (y,\chi)\in S_K\times\Omega_K.$$

Of course, since $f\neq 0$, the set $W:=\text{supp}(Zf)\subseteq S_K\times\Omega_K$ has positive measure. Since the elements in $\widehat{G}$ are continuous, it follows that

$$\sum_{(x,\omega)\in\Lambda}\lambda_{(x,\omega)}\omega(y)\overline{\chi(x)}=0 \text{ for all } (y,\chi)\in W.$$

Now by the structure theorem for compactly generated locally compact abelian Lie groups, G is of the form $G=\mathbb{R}^p\times\mathbb{Z}^q\times\mathbb{T}^r\times F$, where F is a finite group and p,q,r are positive integers or zero. Applying Fubini's theorem yields the existence of elements $k_0\in\mathbb{Z}^q, m_0\in F, \pi_0\in\widehat{\mathbb{T}^r}=\mathbb{Z}^r, \tau_0\in\widehat{F}$ and a measurable set $U_1\subseteq\mathbb{R}^p\times\mathbb{T}^r\times\mathbb{R}^p\times\mathbb{T}^q$ such that $U_1\times\{k_0\}\times\{m_0\}\times\{\pi_0\}\times\{\tau_0\}\subseteq W$ and $\mu_{G\times\widehat{G}}(U_1\times\{k_0\}\times\{m_0\}\times\{\pi_0\}\times\{\tau_0\})>0$. Thus

$$\sum_{(x,\omega)\in\Lambda}\lambda_{(x,\omega)}\omega_1(y_1)\omega_2(k_0)\omega_3(y_3)\omega_4(m_0)\overline{\chi_1(x_1)}\,\overline{\chi_2(x_2)}\,\overline{\pi_0(x_3)}\,\overline{\tau_0(x_4)}=0$$

for all $(y_1,y_3,\chi_1,\chi_2)\in U_1$. To simplify this equation let

$$\tilde{\lambda}_{(x,\omega)}:=\lambda_{(x,\omega)}\omega_2(k_0)\omega_4(m_0)\overline{\pi_0(x_3)}\,\overline{\tau_0(x_4)},\quad (x,\omega)\in\Lambda.$$

Then

$$\sum_{(x,\omega)\in\Lambda}\tilde{\lambda}_{(x,\omega)}\omega_1(y_1)\omega_3(y_3)\overline{\chi_1(x_1)}\,\overline{\chi_2(x_2)}=0 \text{ for all } (y_1,y_3,\chi_1,\chi_2)\in U_1.$$

Now decompose the set Λ into maximal disjoint subsets Λ_m, $1\leq m\leq s$, such that $\omega_1={\omega'}_1$ for all $(x,\omega),(x',\omega')\in\Lambda_m$. This allows us to write the above equation in a more convenient form:

$$\sum_{m=1}^{s}\left[\sum_{(x,\omega)\in\Lambda_m}\tilde{\lambda}_{(x,\omega)}\omega_3(y_3)\overline{\chi_1(x_1)}\,\overline{\chi_2(x_2)}\right]\omega_1(y_1)=0$$

for all $(y_1,y_3,\chi_1,\chi_2)\in U_1$. Fix the elements y_3,χ_1,χ_2 and let the element y_1 vary. By Fubini's theorem, the measurable set

$$U_2:=\{(y_3,\chi_1,\chi_2)\in\mathbb{T}^r\times\mathbb{R}^p\times\mathbb{T}^q:\mu_{\mathbb{R}^p}(\{y_1\in\mathbb{R}^p:(y_1,y_3,\chi_1,\chi_2)\in U_1\})>0\}$$

has positive measure. Now a trigonometric polynomial on $\mathbb{R}^p$ which is zero on a set of positive measure has all coefficients equal to zero. Hence

$$\sum_{(x,\omega)\in\Lambda_m}\tilde{\lambda}_{(x,\omega)}\omega_3(y_3)\overline{\chi_1(x_1)}\,\overline{\chi_2(x_2)}=0 \text{ for all } (y_3,\chi_1,\chi_2)\in U_2, 1\leq m\leq s.$$

For the next step fix any $m \in \{1, \dots, s\}$. Now decompose the set Λ_m as before into maximal disjoint subsets Λ_{m_l}, $1 \leq l \leq s_m$, such that $\omega_3 = \omega'_3$ for all $(x, \omega), (x', \omega') \in \Lambda_{m_l}$. Then we may write the last equation in the form

$$\sum_{l=1}^{s_m} \left[\sum_{(x,\omega)\in\Lambda_{m_l}} \tilde{\lambda}_{(x,\omega)} \overline{\chi_1(x_1)}\, \overline{\chi_2(x_2)} \right] \omega_3(y_3) = 0$$

for all $(y_3, \chi_1, \chi_2) \in U_2$. Fix χ_1 and χ_2 and let y_3 vary. Notice that again by using Fubini's theorem the set

$$U_3 := \{(\chi_1, \chi_2) \in \mathbb{R}^p \times \mathbb{T}^q : \mu_{\mathbb{T}^r}(\{y_3 \in \mathbb{T}^r : (y_3, \chi_1, \chi_2) \in U_2\}) > 0\}$$

is a set of positive measure. A trigonometric polynomial on $\mathbb{T}^r$ which is zero on a set of positive measure can be lifted to a trigonometric polynomial on $\mathbb{R}^r$ which is zero on a set of positive measure. So again the coefficients are equal to zero. Hence

$$\sum_{(x,\omega)\in\Lambda_{m_l}} \tilde{\lambda}_{(x,\omega)} \overline{\chi_1(x_1)}\, \overline{\chi_2(x_2)} = 0 \text{ for all } (\chi_1, \chi_2) \in U_3, 1 \leq l \leq s_m.$$

Now repeating such a step once more with the sets Λ_{m_l}, yields sets $\Lambda_{m_{l_k}}$, $1 \leq k \leq s_{m_l}$, and a set of positive measure $U_4 \subseteq \mathbb{T}^q$. Moreover, the same arguments as before lead to

$$\sum_{(x,\omega)\in\Lambda_{m_{l_k}}} \tilde{\lambda}_{(x,\omega)} \overline{\chi_2(x_2)} = 0 \text{ for all } \chi_2 \in U_4, 1 \leq m \leq s, 1 \leq l \leq s_m, 1 \leq k \leq s_{m_l}.$$

By condition (II), the elements $(x_1, x_2, \omega_1, \omega_3)$, $(x, \omega) \in \Lambda$, are pairwise different. Hence, for all $(x, \omega), (x', \omega') \in \Lambda_{m_{l_k}}$, we obtain $x_2 \neq x'_2$ for each index m_{lk}. The same argument concerning trigonometric polynomials as above yields $\tilde{\lambda}_{(x,\omega)} = 0$ and so $\lambda_{(x,\omega)} = 0$ for all $(x, \omega) \in \Lambda$. This finishes the proof. □

Now we will prove the result for an arbitrary locally compact abelian group G by using the previous proposition. The implication (I) $\Rightarrow$ (II) is an immediate consequence of Theorem 1. Thus we only have to prove that (II) implies (I).

For this let Λ be a finite subset of $K \times A(K, \widehat{G})$ such that the elements $(x, \omega)(G^c \times (\widehat{G})^c)$, $(x, \omega) \in \Lambda$, are pairwise different. Let $p_G : G \times \widehat{G} \to G$ and $p_{\widehat{G}} : G \times \widehat{G} \to \widehat{G}$ denote the projection onto the first and second component, respectively. Let $f \in L^2(G)$, $f \neq 0$, and $(\lambda_{(x,\omega)})_{(x,\omega)\in\Lambda} \subseteq \mathbb{C}$ be such that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)} (\rho_G(x, \omega, 1) f) = 0. \tag{5}$$

The proof will consist of two steps. First the problem will be reduced to locally compact abelian Lie groups and afterwards to compactly generated locally compact abelian Lie groups. Then Proposition 4.1 can be applied.

Recall that locally compact abelian groups are projective limits of Lie groups. Hence as in the proof of Theorem 1, a compact subgroup H of G can be chosen in such a way that G/H is a Lie group and that $\omega \in \widehat{G/H}$ for all $\omega \in p_{\widehat{G}}(\Lambda)$. Now let $\pi : G \to G/H$ denote the quotient homomorphism.

Notice first that since H is compact, the subgroup $\pi(K) = KH/H < G/H$ is a uniform lattice in G/H. One can easily check that

$$A(\pi(K), \widehat{G/H}) = \{\chi \in \widehat{G/H} : \chi \circ \pi \in A(K, \widehat{G})\} < \widehat{G/H}$$

is also a uniform lattice. By the choice of H, it is clear that $\omega \in A(\pi(K), \widehat{G/H})$ for all $\omega \in p_{\widehat{G}}(\Lambda)$. Since $H \subseteq G^c$, we have that $(G/H)^c = \pi(G^c)$. Hence

$$\pi(x)(G/H)^c = \pi(x')(G/H)^c \text{ if and only if } xG^c = x'G^c$$

for all $x, x' \in p_G(\Lambda)$. Observe that $(\widehat{G/H})^c = (\widehat{G})^c \cap \widehat{G/H}$, and so we also obtain that

$$\omega(\widehat{G/H})^c = \omega'(\widehat{G/H})^c \text{ if and only if } \omega(\widehat{G})^c = \omega'(\widehat{G})^c$$

for all $\omega, \omega' \in p_{\widehat{G}}(\Lambda)$. By (II), we can conclude that the elements

$$(\pi(x), \omega)((G/H)^c \times (\widehat{G/H})^c), \quad (x, \omega) \in \Lambda,$$

are pairwise different.

Next we have to construct from f a function f_H on G/H with analogous properties. Let the Haar measures on G and G/H, μ_G and $\mu_{G/H}$, be normalized so that Weil's formula holds, if we take on H the Haar measure μ_H with $\mu_H(H) = 1$. By using Weil's formula and the Cauchy-Schwarz inequality together with the fact that H is compact, we obtain that

$$\begin{aligned} \|f\|_2^2 &= \int_{G/H} \left(\int_H |f(th)|^2 d\mu_H(h) \right) d\mu_{G/H}(\pi(t)) \\ &\geq \int_{G/H} \left(\int_H |f(th)| d\mu_H(h) \right)^2 d\mu_{G/H}(\pi(t)) \\ &\geq \int_{G/H} \left| \int_H f(th) d\mu_H(h) \right|^2 d\mu_{G/H}(\pi(t)). \end{aligned}$$

Now we define a function $f_H : G/H \to \mathbb{C}$ by

$$f_H(\pi(t)) := \int_H f(th) d\mu_H(h), \quad t \in G.$$

As we have just shown, $f_H \in L^2(G/H)$. Without loss of generality we can assume that $f_H \neq 0$ on G/H. By equation (5),

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)} \omega(t) f(xt) = 0 \text{ for } \mu_G\text{-almost all } t \in G.$$

Now observe that the map $G \times H \ni (t, h) \mapsto th \in G$ is continuous and maps $\mu_G \times \mu_H$ onto μ_G. Hence it follows that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)} \omega(t) f(xth) = 0 \text{ for } \mu_G \times \mu_H\text{-almost all } (t, h) \in G \times H.$$

Therefore we obtain that

$$\begin{aligned}
&\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}(\rho_{G/H}(\pi(x),\omega,1)f_H)(\pi(t)) \\
&= \sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}\omega(\pi(t))f_H(\pi(xt)) \\
&= \int_H \left(\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}\omega(t)f(xth) \right) d\mu_H(h) \\
&= 0
\end{aligned}$$

for almost all $\pi(t) \in G/H$. So we can assume that G is a Lie group.

For the last step let L be an open, compactly generated subgroup of G such that $\{x : x \in p_G(\Lambda)\} \subseteq L$ and $f|_L \neq 0$ on L. Then $xx'^{-1} \notin G^c$ implies that $xx'^{-1} \notin L^c$ for all $x, x' \in p_G(\Lambda)$. Recall that $(\widehat{G})^c = \bigcup_{M<G,\, M \text{ open}} \widehat{G/M}$. Thus we can assume that L also fulfills the property that $\omega|_L\overline{\omega'|_L} \notin (\widehat{L})^c$ if $\omega\overline{\omega'} \notin (\widehat{G})^c$ for all $\omega, \omega' \in p_{\widehat{G}}(\Lambda)$. By (II), this shows that the elements

$$(xL^c, \omega|_L(\widehat{L})^c), \quad (x,\omega) \in \Lambda,$$

are pairwise different. Finally, equation (5) implies that

$$\sum_{(x,\omega)\in\Lambda} \lambda_{(x,\omega)}(\rho_L(x,\omega|_L,1)f|_L) = 0 \text{ on } L.$$

Applying Proposition 4.1 yields $\lambda_{(x,\omega)} = 0$ for all $(x,\omega) \in \Lambda$.

5. Some remarks

We finish the paper by adding remarks concerning generalizations and special cases of the theorems.

Remark 5.1. Let G be a locally compact abelian group. Concerning Theorems 1 and 2 it appears to be more natural to consider finite subsets of $H(G)$ instead of finite subsets of $G \times \widehat{G}$, since the Schrödinger representation is defined on $H(G)$. But actually this turns out to be equivalent.

To show this, let Λ be a finite subset of $H(G)$ and let $q : H(G) \to H(G)/\mathbb{T}$ denote the quotient map. First, Remark 2.1 implies that for any two elements (x,ω,z), $(x',\omega',z') \in \Lambda$, we have $(x,\omega,z)H(G)^c \neq (x',\omega',z')H(G)^c$ if and only if $(x,\omega)(G^c \times (\widehat{G})^c) \neq (x',\omega')(G^c \times (\widehat{G})^c)$. Secondly, if $q|_\Lambda$ is injective, then, for each $f \in L^2(G)$, $f \neq 0$, the following conditions are equivalent.

(i) $\{\rho_G(x,\omega,z)f : (x,\omega,z) \in \Lambda\}$ is linearly independent.
(ii) $\{\rho_G(x,\omega,1)f : (x,\omega) \in q(\Lambda)\}$ is linearly independent.

This is an immediate consequence of the definition of the Schrödinger representation. Now it is easy to check that these two statements imply the claim.

Remark 5.2. The uniform lattices in $\mathbb{R}$ are precisely the subgroups of the form $p\mathbb{Z}$, $p \in \mathbb{R}^*$. Hence, for $G = \mathbb{R}$, Theorem 2 covers exactly the situation of Λ being a finite subset of $p\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$, $p \in \mathbb{R}^*$. This does not include all unit lattices in $\mathbb{R}\times\mathbb{R}$. However, generalizing the notion of metaplectic transform to locally compact abelian groups we can prove a corollary of Theorem 2 which covers this case (compare [8, Corollary 4.3.11]).

The question whether a generalization to locally compact abelian groups of Linnell's result holds remains still open, since the set of unit lattices in $\mathbb{R}\times\mathbb{R}$ is a proper subset of the set of discrete subgroups of $\mathbb{R} \times \mathbb{R}$.

Remark 5.3. As mentioned by the referee the case of finite abelian groups "is really the situation one encounters in applications". Let F be a finite abelian group. Then $F^c = F$ and $\widehat{F} = F$. This implies that condition (II) of the conjecture can never be satisfied. Hence for each finite subset Λ of $F \times \widehat{F}$ there exists a function $f \in L^2(F)$, $f \neq 0$, such that the set $\{\rho_F(x, \omega, 1)f : (x, \omega) \in \Lambda\}$ is linearly dependent.

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References

[1] O. Christensen, *Frames containing a Riesz basis and approximation of the frame coefficients using finite-dimensional methods*, J. Math. Anal. Appl. **199** (1996), 256-270.
[2] H.G. Feichtinger and K. Gröchenig, *Gabor frames and time-frequency analysis of distributions*, J. Funct. Anal. **146** (1997), 464-495.
[3] H.G. Feichtinger, T. Strohmer, *Gabor analysis and algorithms: theory and applications*, Birkhäuser, Boston, 1998.
[4] M.A. Rieffel, *Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner*, Adv. in Math. Suppl. Studies **4** (1979), 43-82.
[5] C. Heil, J. Ramanathan and P. Topiwala, *Linear independence of time-frequency translates*, Proc. Amer. Math. Soc. **124** (1996), 2787- 2795.
[6] E. Hewitt and K.A. Ross, *Abstract harmonic analysis I, II*, Springer-Verlag, Berlin/Heidelberg/New York, 1963/1970.
[7] E. Kaniuth and G. Kutyniok, *Zeros of the Zak transform on locally compact abelian groups*, Proc. Amer. Math. Soc. **126** (1998), 3561-3569.
[8] G. Kutyniok, *Time-frequency analysis on locally compact groups*, Ph.D. thesis, University of Paderborn, Germany, 2000.
[9] P.A. Linnell, *Von Neumann algebras and linear independence of translates*, Proc. Amer. Math. Soc. **127** (1999), 3269-3277.

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