THE LOCAL INTEGRABILITY CONDITION FOR WAVELET FRAMES

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Abstract. In this paper we study finitely generated wavelet systems with arbitrary dilation sets. In 2002 Hernández, Labate, and Weiss gave a characterization of when such a system forms a Parseval frame, assuming that a certain hypothesis known as the local integrability condition (LIC) holds. We show that, under some mild regularity assumption on the wavelets, the LIC is solely a density condition on the dilation sets. Using this new interpretation of the LIC, we further discuss when the characterization result holds.

1. Introduction

Frames have turned out to be an essential tool for many emerging applications, since they are robust not only against noise but also against losses (see, for example, [1, 2]). *Parseval frames*, i.e., systems $\{f_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ for which

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 = \|f\|^2 \quad \text{for all } f \in \mathcal{H},$$

enjoy rapidly increasing attention, since these frames are exactly those systems which satisfy the perfect reconstruction formula $f = \sum_{i\in I} \langle f, f_i\rangle f_i$, even though the system may be highly redundant.

One of the most useful types of systems are wavelet systems, and Parseval frame properties of these systems have recently been the focus of a number of papers, e.g., [3, 4, 6, 14, 15, 16]. Most results are concerned with classical wavelet systems where the dilation set has the form $\{a^j : j \in \mathbb{Z}\}$, where $a > 1$, but due to questions arising from sampling theory and concerning perturbation of the dilation sets the necessity of studying wavelet systems with arbitrary dilation sets has occurred.

In this paper we focus on finitely generated wavelet systems of the form

$$\bigcup_{l=1}^{L} \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z}) = \bigcup_{l=1}^{L} \{\tfrac{1}{\sqrt{s}}\psi_l(\tfrac{x}{s} - b_l k) : s \in S_l, k \in \mathbb{Z}\}, \tag{1}$$

where $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ are finitely many sequences of arbitrary dilations, $b_1, \ldots, b_L > 0$, and $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ are wavelets. Recall that a *wavelet* is a function $\psi \in L^2(\mathbb{R})$ satisfying the admissibility condition $\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2/|\xi|\, d\xi < \infty$, which is a necessary

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condition for a wavelet system to form a frame [5]. In [12] (compare also [8] for a correction and some improvements in the situation of classical wavelet systems in higher dimensions) Hernández, Labate, and Weiss gave a characterization of when a system of the form (1) constitutes a Parseval frame for $L^2(\mathbb{R})$, assuming that this system satisfies a certain hypothesis known as the local integrability condition, which is defined as follows.

Definition 1.1. A system of the form (1) satisfies the *local integrability condition (LIC)*, if for all $f \in L^2(\mathbb{R})$ with $\hat{f} \in L^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{f}$ being compact in $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$,

$$I(f) = \sum_{l=1}^{L} \frac{1}{b_l} \sum_{s \in S_l} \sum_{m \in \mathbb{Z}} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + \tfrac{m}{s b_l})|^2 |\hat{\psi}_l(s\xi)|^2 \, d\xi < \infty.$$

The purpose of this paper is to study this rather technical-appearing hypothesis. We will show that, under some mild regularity assumption on the wavelets, the LIC is solely a density condition on the dilation sets. More precisely, it will be proven that the LIC is equivalent to the condition that the dilation sets possess a finite upper density. This condition is very natural, since every wavelet frame of the form (1) must have finite upper density.

In addition to this main result, the techniques we employ include some just recently developed as well as some that are new to the study of wavelets. The notion of density which we use was applied in [13] to derive necessary and sufficient conditions for the existence of wavelet frames. Some variations of this density were employed in [10, 11, 17, 18] to study similar questions. The mild regularity condition on the wavelets will be given in terms of membership of the wavelet in a particular Wiener amalgam space on the group $\mathbb{R}^*$. A different amalgam space (on the affine group) has been recently used in the study of the Homogeneous Approximation Property (HAP) for wavelet systems in [11].

This paper is organized as follows. In Section 2 we introduce the notion of upper and lower density for subsets of $\mathbb{R}^+$ and state some basic properties. Our main results are presented in Section 3. In Section 3.1 we give the definition of an amalgam space on the group $\mathbb{R}^*$. In Section 3.2 we derive an equivalent formulation of the LIC in terms of density conditions (Theorem 3.3). This yields a characterization of finitely generated wavelet Parseval frames with arbitrary dilations provided that the dilation sets possess a finite upper density and the wavelets belong to an amalgam space (Theorem 3.7). This result is stated in Section 3.3. Using the new interpretation of the LIC, in this section we further discuss when the characterization result holds.

2. Density for sequences in $\mathbb{R}^+$

In this section we will derive a notion of density for sequences in $\mathbb{R}^+$ adapted to the geometry of the multiplicative group $\mathbb{R}^+$ in the spirit of the definition of Beurling density on Euclidean space and affine Beurling density on the affine group. Notice that throughout, although S will always denote a sequence of points in $\mathbb{R}^+$ and not merely a subset, for simplicity we will write $S \subseteq \mathbb{R}^+$.

For $h > 0$, we let Q_h denote a fixed increasing, exhaustive family of neighborhoods of the identity element 1 in $\mathbb{R}^+$. For simplicity of computation, we will take

$$Q_h = [e^{-\frac{h}{2}}, e^{\frac{h}{2}}).$$

For $x \in \mathbb{R}^+$, we let xQ_h denote the set Q_h translated via multiplication by x. Let $\mu = \frac{dx}{x}$ denote the Haar measure on $\mathbb{R}^+$. Since μ is invariant under multiplication, we have that

$$\mu(xQ_h) = \mu(Q_h) = \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{dx}{x} = h.$$

Then the upper and lower densities of a sequence in $\mathbb{R}^+$ are defined as follows.

Definition 2.1. Given $S \subseteq \mathbb{R}^+$, the *upper (Beurling) density* of S is defined by

$$\mathcal{D}^+(S) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^+} \frac{\#(S \cap xQ_h)}{h},$$

and the *lower (Beurling) density* of S is

$$\mathcal{D}^-(S) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^+} \frac{\#(S \cap xQ_h)}{h}.$$

If $\mathcal{D}^-(S) = \mathcal{D}^+(S)$, then S has *uniform (Beurling) density*, and we denote this density by $\mathcal{D}(S)$.

Recall that the wavelet systems we are considering, of the form given in (1), possess finitely many dilation sets. In order to equip such a system with a single density, we will consider the disjoint union of the single dilation sets.

Remark 2.2. Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$. We will use the notation $S = \bigcup_{l=1}^{L} S_l$ to denote the *disjoint union* of these sequences. In particular, if each S_l is indexed as $S_l = \{s_{kl}\}_{k\in\mathbb{N}}$, then S is the sequence $S = \{s_{11}, \ldots, s_{1L}, s_{21}, \ldots, s_{2L}, \ldots\}$. We have the following relation between the density of the single sequences and the density of their disjoint union:

$$\sum_{l=1}^{L} \mathcal{D}^-(S_l) \leq \mathcal{D}^-(S) \leq \mathcal{D}^+(S) \leq \sum_{l=1}^{L} \mathcal{D}^+(S_l).$$

These inequalities may be strict, e.g., consider $S_1 = \{e^m : m \in \mathbb{Z}, m \geq 0\}$ and $S_2 = \{e^m : m \in \mathbb{Z}, m < 0\}$, where $L = 2$.

We obtain the following useful reinterpretation of finite upper density of a single sequence.

Proposition 2.3. *Let $S \subseteq \mathbb{R}^+$. Then the following conditions are equivalent.*

(i) $\mathcal{D}^+(S) < \infty$.
(ii) *There exists an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu(I) < \infty$ such that* $\sup_{x\in\mathbb{R}^+} \#(S \cap xI) < \infty$.
(iii) *For every interval $I \subseteq \mathbb{R}^+$ with $0 < \mu(I) < \infty$, we have* $\sup_{x\in\mathbb{R}^+} \#(S \cap xI) < \infty$.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) are trivial.

In the following we prove (ii) $\Rightarrow$ (i), (iii). Suppose there exists an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu(I) < \infty$ and some constant $N < \infty$ with $\#(S \cap xI) < N$ for all $x \in \mathbb{R}^+$. Let J be another interval in $\mathbb{R}^+$ with $0 < \mu(J) < \infty$. If there exists $y \in \mathbb{R}^+$ with $yJ \subseteq I$, then $\#(S \cap xJ) < N$ for all $x \in \mathbb{R}^+$. On the other hand, if there exists $y \in \mathbb{R}^+$ with $yI \subseteq J$, then $\mu(J) = r\mu(I)$ for some $r \geq 1$, and J is covered by a union of at most $r+1$ sets of the form xI. Consequently,

$$\sup_{x\in\mathbb{R}^+} \#(S \cap xJ) \leq (r+1) \sup_{x\in\mathbb{R}^+} \#(S \cap xI) \leq (r+1)N.$$

Thus statement (iii) holds. Further,

$$\mathcal{D}^+(S) \leq \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^+} \#(S \cap xQ_{r\mu(I)})}{r\mu(I)} \leq \limsup_{r\to\infty} \frac{(r+1)N}{r\mu(I)} = \frac{N}{\mu(I)} < \infty,$$

so statement (i) holds as well. □

The following immediate consequence will be a useful tool for reducing statements about density for disjoint unions of sequences in $\mathbb{R}^+$ to density of the single sequences.

Proposition 2.4. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$, and let $S = \bigcup_{l=1}^{L} S_l$. Then the following conditions are equivalent.*

(i) *We have $\mathcal{D}^+(S) < \infty$.*
(ii) *For all $1 \leq l \leq L$, we have $\mathcal{D}^+(S_l) < \infty$.*

3. Density and the LIC

In this section we will show that the LIC is equivalent to a density condition on the sets of dilations provided that the Fourier transform of the generating wavelets are contained in a particular amalgam space.

3.1. **Amalgam Spaces.** An amalgam space combines a local criterion for membership with a global criterion. For our purposes, we will need only the following particular amalgam space on the group $\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. The backbone of the definition of the discrete-type norm of this amalgam space is the choice of a collection of subsets of $\mathbb{R}^*$. However, it can be shown that in fact the definition is not dependent on a particular choice; the only requirement is that the characteristic functions of the subsets belonging to this collection form a bounded uniform partition of unity (BUPU) in the terminology of [7]. In our case, for each $h > 0$, we let $K_h := -Q_h \cup Q_h$, and we will use the notation $K_h(x) = -xQ_h \cup xQ_h$ for $x \in \mathbb{R}^*$. It is easily checked that $\{K_1(e^k) : k \in \mathbb{Z}\}$ provides us with a tiling of $\mathbb{R}^*$; hence, $\{\chi_{K_1(e^k)}\}_{k\in\mathbb{Z}}$ forms a BUPU. Using this particular tiling we can define the amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$ on the group $\mathbb{R}^*$ as follows.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{C}$ belongs to the *amalgam space* $W_{\mathbb{R}^*}(L^\infty, L^2)$ if

$$\|f\|_{W_{\mathbb{R}^*}(L^\infty,L^2)} = \left(\sum_{k\in\mathbb{Z}} \operatorname{esssup}_{x\in K_1(e^k)} |f(x)|^2 \right)^{\frac{1}{2}} < \infty.$$

For an expository introduction to amalgam spaces with extensive references to the original literature, we refer to [9].

The following lemma shows that the consideration of wavelets whose Fourier transform is contained in this amalgam space is by no means restrictive, and is even natural. Specifically, a mild decay condition on $\hat{\psi}$ suffices to ensure that $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

Lemma 3.2. *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Suppose that there exist $a, b, \alpha, \beta > 0$ such that $|\hat{\psi}(\xi)| \le a|\xi|^\alpha$ as $|\xi| \to 0$ and $|\hat{\psi}(\xi)| \le b|\xi|^{-\beta}$ as $|\xi| \to \infty$. Then $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.*

Proof. Let $0 < \omega \le \Omega < \infty$ be such that $|\hat{\psi}(\xi)| \le a|\xi|^\alpha$ for all $|\xi| \le \omega$ and $|\hat{\psi}(\xi)| \le b|\xi|^{-\beta}$ for all $|\xi| \ge \Omega$. Since $\psi \in L^1(\mathbb{R})$, hence $\hat{\psi} \in C(\mathbb{R})$, there exists $M < \infty$ such that $|\hat{\psi}(\xi)| \le M$ for all $|\xi| \in [e^{-1}\omega, e\Omega]$. Then,

$$\begin{aligned}
&\|\hat{\psi}\|^2_{W_{\mathbb{R}^*}(L^\infty, L^2)} \\
&= \sum_{k \in \mathbb{Z}} \sup_{\xi \in K_1(e^k)} |\hat{\psi}(\xi)|^2 \\
&\le \sum_{k \in \mathbb{Z}, k \le \ln\omega - \frac{1}{2}} \sup_{\xi \in K_1(e^k)} |\hat{\psi}(\xi)|^2 + \sum_{k=\lceil \ln\omega - \frac{1}{2}\rceil}^{\lfloor \ln\Omega + \frac{1}{2}\rfloor} \sup_{\xi \in K_1(e^k)} |\hat{\psi}(\xi)|^2 + \sum_{k \in \mathbb{Z}, k \ge \ln\Omega + \frac{1}{2}} \sup_{\xi \in K_1(e^k)} |\hat{\psi}(\xi)|^2 \\
&\le a^2 \sum_{k=-\infty}^{\lfloor \ln\omega - \frac{1}{2}\rfloor} |e^{k+\frac{1}{2}}|^{2\alpha} + (\ln\Omega - \ln\omega + 2)M^2 + b^2 \sum_{k=\lceil \ln\Omega + \frac{1}{2}\rceil}^{\infty} |e^{k-\frac{1}{2}}|^{-2\beta} < \infty. \quad \square
\end{aligned}$$

Let $\psi \in L^2(\mathbb{R})$ be a wavelet. If $\psi \in L^1(\mathbb{R})$, then $\hat{\psi}(0) = 0$. Thus, if in addition a wavelet ψ possesses a Fourier transform with polynomial decay towards zero and infinity, then its Fourier transform is contained in $W_{\mathbb{R}^*}(L^\infty, L^2)$.

3.2. **A Density Version of the LIC.** Now we turn to the interpretation of the LIC (see Definition 1.1) in terms of the density of the dilation sets. Our main result gives an equivalent formulation of the LIC in terms of density conditions.

Theorem 3.3. *Let $S_1, \dots, S_L \subseteq \mathbb{R}^+$, let $S = \bigcup_{l=1}^L S_l$, and let $b_1 \dots, b_L > 0$ be given. Then the following conditions are equivalent.*

(i) *We have $\mathcal{D}^+(S) < \infty$.*
(ii) *For all $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \dots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$, the wavelet system $\bigcup_{l=1}^L \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ satisfies the LIC.*

We will break its proof into several parts to improve clarity. First we derive an easy equivalent formulation of the LIC better suited to our purposes.

Lemma 3.4. *Let $S_1, \dots, S_L \subseteq \mathbb{R}^+$, $b_1 \dots, b_L > 0$, and $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ be given. Then the following conditions are equivalent.*

(i) *The system $\bigcup_{l=1}^L \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ satisfies the LIC.*

(ii) *For all* $l = 1, \ldots, L$ *and* $h > 0$,

$$I_l(h) = \frac{1}{b_l} \sum_{s \in S_l} \frac{1}{s} \sum_{m \in \mathbb{Z}} \int_{K_h(s) \cap (K_h(s) - \frac{m}{b_l})} |\hat{\psi}_l(\xi)|^2 \, d\xi < \infty.$$

Proof. (i) $\Rightarrow$ (ii). Let $I(f)$ be defined as in Definition 1.1. Suppose that (i) holds, i.e., $I(f) < \infty$ for all $f \in L^2(\mathbb{R})$ such that $\hat{f} \in L^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{f}$ is compact in $\mathbb{R}^*$. Then choosing $f \in L^2(\mathbb{R})$ with $\hat{f} = \chi_{K_h}$ and observing that each of the terms I_l, $l = 1, \ldots, L$, is positive, implies (ii).

(ii) $\Rightarrow$ (i). For $l = 1, \ldots, L$ and $h > 0$, we first notice that

$$I_l(h) = \frac{1}{b_l} \sum_{s \in S_l} \sum_{m \in \mathbb{Z}} \int_{K_h} \chi_{K_h}(\xi + \tfrac{m}{s b_l}) \, |\hat{\psi}_l(s\xi)|^2 \, d\xi. \tag{2}$$

Now let $f \in L^2(\mathbb{R})$ be such that $\hat{f} \in L^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{f}$ is compact in $\mathbb{R}^*$. Then there exists $M < \infty$ and a compact set $K \subseteq \mathbb{R}^*$ such that $|\hat{f}(\xi)| \le M \chi_K(\xi)$ for almost every $\xi \in \mathbb{R}$. Since $(K_h)_{h>0}$ is an exhaustive sequence of compact sets in $\mathbb{R}^*$, there exists $h > 0$ such that $K \subseteq K_h$. By (ii) and (2), this yields

$$\begin{aligned} I(f) &= \sum_{l=1}^{L} \frac{1}{b_l} \sum_{s \in S_l} \sum_{m \in \mathbb{Z}} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + \tfrac{m}{s b_l})|^2 \, |\hat{\psi}_l(s\xi)|^2 \, d\xi \\ &\le M^2 \sum_{l=1}^{L} \frac{1}{b_l} \sum_{s \in S_l} \sum_{m \in \mathbb{Z}} \int_K \chi_K(\xi + \tfrac{m}{s b_l}) \, |\hat{\psi}_l(s\xi)|^2 \, d\xi \\ &\le M^2 \sum_{l=1}^{L} I_l(h) \; < \; \infty. \end{aligned}$$

Thus (i) is satisfied. □

The following lemma establishes a relation between density, the amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$, and a Littlewood–Paley–type inequality.

Lemma 3.5. *Let* $S \subseteq \mathbb{R}^+$ *with* $\mathcal{D}^+(S) < \infty$ *be given, and let* $\psi \in L^2(\mathbb{R})$ *with* $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ *be given. Then there exists* $B < \infty$ *such that*

$$\sum_{s \in S} |\hat{\psi}(s\xi)|^2 \le B \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

Proof. For each $k \in \mathbb{Z}$, set $c_k = \operatorname{esssup}_{\xi \in K_1(e^k)} |\hat{\psi}(\xi)|^2$. Then we have

$$|\hat{\psi}(\xi)|^2 \le \sum_{k \in \mathbb{Z}} c_k \chi_{K_1(e^k)}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R} \tag{3}$$

and

$$\sum_{k \in \mathbb{Z}} c_k = \|\hat{\psi}\|^2_{W_{\mathbb{R}^*}(L^\infty, L^2)}. \tag{4}$$

Since $S \subseteq \mathbb{R}^+$, equation (3) yields

$$\begin{aligned}
\operatorname{esssup}_{\xi\in\mathbb{R}^*} \sum_{s\in S} |\hat{\psi}(s\xi)|^2 &\leq \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S}\sum_{k\in\mathbb{Z}} c_k\, \chi_{K_1(e^k)}(s\xi) \\
&\leq \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S} \chi_{K_1(\xi^{-1}e^k)}(s) \\
&= \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S} \chi_{K_1(\xi)}(s) \\
&= \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^+} \#(S\cap \xi Q_1).
\end{aligned}$$

Since $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ and $\mathcal{D}^+(S) < \infty$, the last quantity is a finite constant by equation (4) and Proposition 2.3. □

In the following lemma, by using a sequence in $\mathbb{R}^+$, we explicitly construct functions whose Fourier transform is contained in $W_{\mathbb{R}^*}(L^\infty, L^2)$.

Lemma 3.6. *Let $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be such that the sets y_nQ_1, $n \in \mathbb{N}$, are mutually disjoint.*

(i) *Suppose that $y_n \to 0$ as $n \to \infty$. Then the function $\psi \in L^2(\mathbb{R})$ defined by*

$$\hat{\psi} = \sum_{n\in\mathbb{N}} \frac{1}{n}\, \chi_{y_nQ_1}$$

satisfies $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

(ii) *Suppose that $y_n \to \infty$ as $n \to \infty$. Then the function $\psi \in L^2(\mathbb{R})$ defined by*

$$\hat{\psi} = \sum_{n\in\mathbb{N}} \frac{1}{n\sqrt{y_n}}\, \chi_{y_nQ_1}$$

satisfies $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

Proof. (i) Suppose that $y_n \to 0$ as $n \to \infty$. It is easy to check that $\hat{\psi} \in L^2(\mathbb{R})$, hence $\psi \in L^2(\mathbb{R})$. We next observe that for each $k \in \mathbb{Z}$ and $x \in \mathbb{R}^+$, we have $e^kQ_1 \cap xQ_1 \neq \emptyset$ if and only if $\ln x - 1 \leq k \leq \ln x + 1$. Therefore we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2 &= \sum_{n\in\mathbb{N}} \frac{1}{n^2} \sum_{k\in\mathbb{Z}} \sup_{\xi\in e^kQ_1} \chi_{y_nQ_1}(\xi) \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n^2}\, \#\{k\in\mathbb{Z} : e^kQ_1 \cap y_nQ_1 \neq \emptyset\} \\
&\leq 3\sum_{n\in\mathbb{N}} \frac{1}{n^2} \;<\; \infty.
\end{aligned}$$

(ii) Now suppose that $y_n \to \infty$ as $n \to \infty$. The slightly different definition of ψ ensures that also in this case $\hat{\psi} \in L^2(\mathbb{R})$, hence $\psi \in L^2(\mathbb{R})$. Then $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ can be proven in a similar way as in part (i). □

Now we are prepared to prove Theorem 3.3.

Proof of Theorem 3.3. (i) $\Rightarrow$ (ii). We suppose that $\mathcal{D}^+(S) < \infty$, which by Proposition 2.4 implies that each $\mathcal{D}^+(S_l) < \infty$, $l = 1, \ldots, L$. For arbitrary $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \ldots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$, we have to show that $\bigcup_{l=1}^L \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ satisfies the LIC. As observed in Lemma 3.4, it suffices to prove that

$$I_l(h) = \frac{1}{b_l} \sum_{s \in S_l} \frac{1}{s} \sum_{m \in \mathbb{Z}} \int_{K_h(s) \cap (K_h(s) - \frac{m}{b_l})} |\hat{\psi}_l(\xi)|^2 \, d\xi < \infty, \tag{5}$$

for all $l = 1, \ldots, L$ and $h > 0$. For this, fix $h > 0$, $\psi \in L^2(\mathbb{R})$ with $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$, and consider some $l \in \{1, \ldots, L\}$. For the sake of brevity, we set $I(h) = I_l(h)$, $S = S_l$, and $b = b_l$. We decompose $I(h)$ by

$$I(h) = I_1(h) + I_2(h),$$

where

$$I_1(h) = \frac{1}{b} \sum_{s \in S} \frac{1}{s} \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi$$

and

$$I_2(h) = \frac{1}{b} \sum_{s \in S} \frac{1}{s} \sum_{m \in \mathbb{Z} \setminus \{0\}} \int_{K_h(s) \cap (K_h(s) - \frac{m}{b})} |\hat{\psi}(\xi)|^2 \, d\xi.$$

First, we study $I_1(h)$. By Lemma 3.5, there exists some $B < \infty$ such that

$$\sum_{s \in S} |\hat{\psi}(s\xi)|^2 \le B \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Therefore,

$$I_1(h) = \frac{1}{b} \sum_{s \in S} \frac{1}{s} \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi = \frac{1}{b} \int_{K_h} \sum_{s \in S} |\hat{\psi}(s\xi)|^2 \, d\xi \le \frac{1}{b} B |K_h| < \infty. \tag{6}$$

Secondly, we show that $I_2(h)$ is finite. Let $s \in S$ be fixed. We observe that if $sQ_h \cap (sQ_h - \frac{m}{b}) \neq \emptyset$ then

$$se^{-\frac{h}{2}} \le se^{\frac{h}{2}} - \tfrac{m}{b} \quad \text{and} \quad se^{\frac{h}{2}} \ge se^{-\frac{h}{2}} - \tfrac{m}{b}.$$

This is equivalent to

$$-sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \le m \le sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}).$$

Since $K_h(s) = -sQ_h \cap sQ_h$, for each $s \in S$ there exist at most $3(2sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) + 1)$ integers m such that $K_h(s) \cap (K_h(s) - \frac{m}{b}) \neq \emptyset$. Recall that in this case we only consider $m \in \mathbb{Z} \setminus \{0\}$. Therefore there exists $\epsilon > 0$ such that $K_h(s) \cap (K_h(s) - \frac{m}{b}) = \emptyset$ for all $s \in S$ with $s < \epsilon$ and $m \in \mathbb{Z} \setminus \{0\}$. This shows that we only need to consider those $s \in S$ with $s \ge \epsilon$. Then there exists $C < \infty$ such that

$$3(2sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) + 1) \le Cs(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \text{ for all } s \in S,\ s \ge \epsilon.$$

Using these observations, we obtain

$$\begin{aligned} I_2(h) &\leq \frac{1}{b}\sum_{s\in S}\frac{1}{s}Cs(e^{\frac{h}{2}}-e^{-\frac{h}{2}})\int_{K_h(s)}|\hat{\psi}(\xi)|^2\,d\xi \\ &= \frac{C}{b}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})\sum_{s\in S}\int_{K_h(s)}|\hat{\psi}(\xi)|^2\,d\xi. \end{aligned} \tag{7}$$

It follows easily from $\mathcal{D}^+(S)<\infty$ that there exists $N<\infty$ such that

$$\#\{s\in S : x\in K_h(s)\}\leq N \quad \text{for all } x\in\mathbb{R}.$$

Continuing equation (7), we obtain

$$I_2(h)\leq\frac{C}{b}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})\sum_{s\in S}\int_{K_h(s)}|\hat{\psi}(\xi)|^2\,d\xi\leq\frac{C}{b}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})\,N\,\|\hat{\psi}\|_2^2<\infty. \tag{8}$$

Combining the estimates (6) and (8) yields (5). Thus (ii) holds.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Towards a contradiction assume that we have $\mathcal{D}^+(S)=\infty$. By Proposition 2.4, there exists $l_0\in\{1,\dots,L\}$ with $\mathcal{D}^+(S_{l_0})=\infty$. Thus, by Lemma 3.4, it suffices to show that there exists $\psi\in L^2(\mathbb{R})$ with $\hat{\psi}\in W_{\mathbb{R}^*}(L^\infty,L^2)$ such that for some $h>0$,

$$I_{l_0}(h)=\frac{1}{b_{l_0}}\sum_{s\in S_{l_0}}\frac{1}{s}\sum_{m\in\mathbb{Z}}\int_{K_h(s)\cap(K_h(s)-\frac{m}{b_{l_0}})}|\hat{\psi}(\xi)|^2\,d\xi=\infty. \tag{9}$$

To simplify notation we set $I(h)=I_{l_0}(h)$, $S=S_{l_0}$, and $b=b_{l_0}$. Proposition 2.3 implies the existence of sequences $(y_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}^+$ and $(S_n)_{n\in\mathbb{N}}$ with $S_n\subseteq S$ satisfying that $\#S_n\geq n$ and $S_n\subseteq y_nQ_1$.

If there exists $y\in\mathbb{R}^+$ and $h>0$ with $\#(S\cap yQ_h)=\infty$, then choosing $\psi\in L^2(\mathbb{R})$ by $\hat{\psi}=\chi_{yQ_hK_h}\in W_{\mathbb{R}^*}(L^\infty,L^2)$ yields

$$\begin{aligned} I(h) &= \frac{1}{b}\sum_{s\in S}\frac{1}{s}\sum_{m\in\mathbb{Z}}\int_{K_h(s)\cap(K_h(s)-\frac{m}{b})}|\hat{\psi}(\xi)|^2\,d\xi \\ &\geq \frac{1}{b}\sum_{s\in S\cap yQ_h}\frac{1}{s}\int_{K_h(s)}\chi_{yQ_hK_h}(\xi)\,d\xi \\ &= \frac{1}{b}\sum_{s\in S\cap yQ_h}\frac{1}{s}s|K_h|=\infty. \end{aligned}$$

Otherwise we remark that, by restricting $(y_n)_{n\in\mathbb{N}}$ to a subsequence if necessary, we have either $y_n\to 0$ or $y_n\to\infty$ as $n\to\infty$. Moreover, without loss of generality we may assume that the sets y_nQ_1, $n\in\mathbb{N}$, are mutually disjoint, by choosing again an appropriate subsequence if necessary.

First assume that $y_n\to 0$ as $n\to\infty$. Then we define the function $\psi\in L^2(\mathbb{R})$ by

$$\hat{\psi}=\sum_{n\in\mathbb{N}}\frac{1}{n}\,\chi_{y_nQ_1}.$$

Lemma 3.6(i) implies that $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$. We now choose $h = 2$. Then,

$$\begin{aligned} I(2) &= \frac{1}{b}\sum_{s\in S}\frac{1}{s}\sum_{m\in\mathbb{Z}}\int_{K_2(s)\cap(K_2(s)-\frac{m}{b})}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\,\chi_{y_nQ_1}(\xi)\,d\xi \\ &\geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{1}{s}\sum_{m\in\mathbb{Z}}|sQ_2\cap(sQ_2-\tfrac{m}{b})\cap y_nQ_1|. \end{aligned} \tag{10}$$

Fix $n \in \mathbb{N}$. Since $S_n \subseteq y_nQ_1$, it follows that for each $s \in S_n$,

$$y_nQ_1 = [y_ne^{-\frac{1}{2}}, y_ne^{\frac{1}{2}}) \subseteq [se^{-\frac{1}{2}}e^{-\frac{1}{2}}, se^{\frac{1}{2}}e^{\frac{1}{2}}) = [se^{-1}, se) = sQ_2. \tag{11}$$

This implies that $\frac{y_n}{s}Q_1 \subseteq Q_2$, and hence an easy computation shows that $\frac{y_n}{s} \in [e^{-\frac{1}{2}}, e^{\frac{1}{2}})$. Thus

$$|Q_2 \cap \tfrac{y_n}{s}Q_1| \geq 1 - e^{-1} \quad \text{for all } n \in \mathbb{N},\ s \in S_n. \tag{12}$$

Therefore, employing (11) and (12), we can continue the computation in (10) to obtain

$$\begin{aligned} I(2) &\geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{1}{s}\sum_{m\in\mathbb{Z}}|sQ_2\cap(sQ_2-\tfrac{m}{b})\cap y_nQ_1| \\ &\geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{1}{s}|sQ_2\cap y_nQ_1| \\ &= \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}|Q_2\cap\tfrac{y_n}{s}Q_1| \\ &\geq \frac{1-e^{-1}}{b}\sum_{n\in\mathbb{N}}\frac{1}{n} = \infty. \end{aligned}$$

This settles (9) for $h = 2$.

Secondly, assume that $y_n \to \infty$ as $n \to \infty$. In this case we define $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi} = \sum_{n\in\mathbb{N}}\frac{1}{n\sqrt{y_n}}\,\chi_{y_nQ_1}.$$

Lemma 3.6(ii) implies that $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$. We observe that $sQ_2 \subseteq sQ_4 - \frac{m}{b}$ if and only if

$$-sb(e^{-1} - e^{-2}) \leq m \leq sb(e^2 - e).$$

Thus there exist at least $sb(e^2 - e + e^{-1} - e^{-2}) = sbC'$ values of m for which this containment is true. Choosing $h = 4$ and using (11) yields

$$\begin{aligned}
I(4) &\geq \frac{1}{b}\sum_{n\in\mathbb{N}} \frac{1}{n^2 y_n} \sum_{s\in S_n} \frac{1}{s} \sum_{m\in\mathbb{Z}} |sQ_4 \cap (sQ_4 - \tfrac{m}{b}) \cap y_n Q_1| \\
&\geq \frac{1}{b}\sum_{n\in\mathbb{N}} \frac{1}{n^2 y_n} \sum_{s\in S_n} \frac{1}{s} sbC'|y_n Q_1| \\
&= C'|Q_1| \sum_{n\in\mathbb{N}} \frac{1}{n^2 y_n} \sum_{s\in S_n} y_n \\
&\geq C'|Q_1| \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty.
\end{aligned}$$

This settles (9) for $h = 4$.

Hence (ii) is not satisfied, a contradiction. □

3.3. **A Characterization of Wavelet Parseval Frames.** The equivalent formulation of the LIC in terms of density conditions yields the following characterization result for finitely generated wavelet Parseval frames with arbitrary dilation sets.

Theorem 3.7. *Let $S_1, \dots, S_L \subseteq \mathbb{R}^+$, and let $S = \bigcup_{l=1}^L S_l$. Suppose that $\mathcal{D}^+(S) < \infty$. Then for all $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \dots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$ and $b_1, \dots, b_L \in \mathbb{R}^+$, the following conditions are equivalent.*

(i) *$\bigcup_{l=1}^L \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ is a Parseval frame for $L^2(\mathbb{R})$.*

(ii) *For each $\alpha \in \bigcup_{l=1}^L \bigcup_{s\in S_l} \frac{1}{b_l s}\mathbb{Z}$, where $\mathcal{P}_\alpha = \{(l,s) \in \{1,\dots,L\} \times S_l : b_l s\alpha \in \mathbb{Z}\}$, we have*

$$\sum_{(l,s)\in\mathcal{P}_\alpha} \frac{1}{b_l} \overline{\hat{\psi}_l(s\xi)} \hat{\psi}_l(s(\xi+\alpha)) = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Proof. The claim follows immediately from Theorem 3.3 and [12, Thm. 2.1]. □

At last, we show that the hypothesis of finite upper density is not at all restrictive.

Proposition 3.8. *Let $S_1, \dots, S_L \subseteq \mathbb{R}^+$, and let $S = \bigcup_{l=1}^L S_l$. Further, let $b_1, \dots, b_L > 0$ and $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ be given. Then Theorem 3.7(i) implies $\mathcal{D}^+(S) < \infty$, and if, in addition, $\psi_1, \dots, \psi_L \in L^1(\mathbb{R})$, then Theorem 3.7(ii) implies $\mathcal{D}^+(S) < \infty$.*

Proof. First suppose that Theorem 3.7(i) holds, i.e., that $\bigcup_{l=1}^L \mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ is a Parseval frame for $L^2(\mathbb{R})$. Then, in particular, for each $l = 1, \dots, L$ the wavelet system $\mathcal{W}(\psi_l, S_l \times b_l\mathbb{Z})$ is a Bessel sequence, i.e., it possesses an upper frame bound. Now [10, Thm. 1.1(a)] implies that $\mathcal{D}^+(S_l \times b_l\mathbb{Z}) < \infty$ for all $l = 1, \dots, L$. A simple computation shows that this implies $\mathcal{D}^+(S_l) < \infty$ for all $l = 1, \dots, L$. The application of Proposition 2.4 then proves the first claim.

Secondly, suppose that Theorem 3.7(ii) holds. Notice that $\mathcal{P}_0 = \{(l,s) \in \{1,\ldots,L\} \times S_l\}$. Hence in the special case $\alpha = 0$ we obtain

$$\sum_{l=1}^{L} \frac{1}{b_l} \sum_{s \in S_l} |\hat{\psi}_l(s\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Towards a contradiction assume that there exists $l_0 \in \{1,\ldots,L\}$ with $\mathcal{D}^+(S_{l_0}) = \infty$. There exist some $\xi_0 \in \mathbb{R}$ and $h_0 > 0$ such that $|\hat{\psi}_{l_0}(\xi)|^2 \geq \delta > 0$ for every $\xi \in \xi_0 Q_{h_0}$. Applying Proposition 2.3, for each $n \in \mathbb{N}$, there exists some $y_n \in \mathbb{R}$ with

$$\#(S_{l_0} \cap y_n Q_{h_0}) \geq n.$$

Hence,

$$\sum_{s \in S_{l_0}} |\hat{\psi}_{l_0}(s(\xi_0 y_n^{-1}))|^2 \geq \sum_{s \in S_{l_0} \cap y_n Q_{h_0}} |\hat{\psi}_{l_0}(s(\xi_0 y_n^{-1}))|^2 \geq \delta\, n,$$

a contradiction. Thus $\mathcal{D}^+(S_l) < \infty$ for all $l = 1,\ldots,L$. Proposition 2.4 then settles the claim. □

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