The geometry of sets of parameters of wave packet frames

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Abstract

We study wave packet systems $\mathcal{WP}(\psi, \mathcal{M})$; that is, countable collections of dilations, translations, and modulations of a single function $\psi \in L^2(\mathbb{R})$. The parameters of these unitary actions form a discrete subset $\mathcal{M} \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. We introduce analogues of the notion of Beurling density, adapted to the geometry of discrete subsets of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, and notions of lower and upper dimensions associated with these densities. Our goal is to describe completeness properties of wave packet systems via geometric properties of the sets of their parameters. In particular, we show necessary conditions for $\mathcal{WP}(\psi, \mathcal{M})$ to be a Bessel system, and we construct multiple examples of non-standard wave packet frames with prescribed dimensions.

Key words: Beurling density, dimension, frame, Gabor system, wavelet, wave packet system
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1 Introduction

Let $\mathcal{M} \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ and $\psi \in L^2(\mathbb{R})$. The *wave packet system* $\mathcal{WP}(\psi, \mathcal{M})$ is the collection

$$\{\sqrt{x}e^{2\pi i(xt-y)z}\psi(xt-y) : (x,y,z) \in \mathcal{M}\}. \tag{1}$$

The elements of $\mathcal{M}$ are called the *parameters* of the wave packet system. Let I be an index set. A collection $\{x_i : i \in I\}$ of vectors in the Hilbert space H is said to be a *frame* if there exist positive constants A and B such that for each $x \in H$,

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2. \tag{2}$$

Positive constants A and B for which (2) holds are called lower and upper frame bounds for $\{x_i : i \in I\}$. When the second inequality in (2) holds, but not necessarily the first inequality, then we call $\{x_i : i \in I\}$ a Bessel system. Finally, a Riesz basis for H is the image of an orthonormal basis for H under an invertible bounded linear operator on H. A wave packet system which is a frame [resp. orthonormal basis, Riesz basis] for $L^2(\mathbb{R})$ will be called a wave packet frame [resp. orthonormal basis, Riesz basis], and a wave packet system which is a Bessel system will be called a wave packet Bessel system.

There are at least three well-studied special cases of wave packet systems — Gabor systems, wavelet systems, and the Fourier transform of wavelet systems. For example, when $\mathcal{M}_1 = \{(1, y, z) : y, z \in \mathbb{Z}\}$, one obtains (the Fourier transform of) a Gabor system $\mathcal{WP}(\psi, \mathcal{M}_1) = \{e^{-2\pi iyz}e^{2\pi izt}\psi(t-y) : y, z \in \mathbb{Z}\}$. When $\mathcal{M}_2 = \{(2^j, k, 0) : j, k \in \mathbb{Z}\}$, one obtains a wavelet system $\mathcal{WP}(\psi, \mathcal{M}_2) = \{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$. Similarly, when $\mathcal{M}_3 = \{(2^j, 0, k) : j, k \in \mathbb{Z}\}$, $\mathcal{WP}(\psi, \mathcal{M}_3)$ is the Fourier transform of a wavelet system. More general wave packet systems have recently been successfully applied to problems in harmonic analysis and operator theory [7,13–15]. In addition, there are some interesting partial results aimed at understanding wavelet systems and Gabor systems as limiting cases of more general wave packet systems [6].

In this paper, we focus on a different aspect of wave packet systems. Notice that $\mathcal{M}_1$ is a lattice of rank 2 in $\mathbb{R}^3$; that is, it is the image of $\mathbb{Z}^3$ under a linear transformation of rank 2. The sets $\mathcal{M}_2' = \{(\ln x, y, z) : (x, y, z) \in \mathcal{M}_2\}$ and $\mathcal{M}_3' = \{(\ln x, y, z) : (x, y, z) \in \mathcal{M}_3\}$ are also lattices of rank 2. In addition,

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it is known that sets of parameters consisting of dilations, translations or modulations alone will yield neither frames nor Riesz bases for $L^2(\mathbb{R})$ [3,8,18]. It is also known (and a consequence of the work in this paper) that for $\mathcal{M}_4 = \{(2^j, k, l) : (j, k, l) \in \mathbb{Z}^3\}$, $\mathcal{WP}(\psi, \mathcal{M}_4)$ is not a Bessel system, unless $\psi = 0$. In light of this, it is natural to ask whether all sets of parameters of wave packet systems that form, say, frames for $L^2(\mathbb{R})$ must be two-dimensional in some sense. This is the question that we address in this paper.

There have been several results which can be interpreted in terms of wave packet systems and their parameters. A classical theorem of Wiener states that for $f \in L^2(\mathbb{R})$, $\{f(x+c) : c \in \mathbb{R}\}$ is complete in $L^2(\mathbb{R})$ if and only if $\hat{f} \neq 0$ almost everywhere. Several authors [1,17,19] have studied those sets $\Lambda \subset \mathbb{R}$ such that $\{f(x+\lambda) : \lambda \in \Lambda\}$ is complete in $L^2(\mathbb{R})$, which would be a first step in constructing Schauder bases for $L^2(\mathbb{R})$ consisting of translates of a single function. It remains open whether one can construct Schauder bases for $L^2(\mathbb{R})$ using only translations, dilations or modulations of a single function [18]. Interpreted in view of wave packet systems, one could say that the research program is to determine whether wave packet systems that have "almost" one-dimensional parameters — for example, perturbations of $\{1\} \times \mathbb{Z} \times \{0\}$ — can form bases for $L^2(\mathbb{R})$. In this paper, we will show that there are wave packet systems that are arbitrarily close to being one dimensional (in a sense that will be made precise in the paper) which are orthonormal bases for $L^2(\mathbb{R})$.

Another result that is related to the research in this paper was obtained in [4]. In that paper, it is shown that for $\Lambda \subset \mathbb{R}^+$ and $\mathcal{M} = \{(x, y, 0) : x \in \Lambda, y \in \mathbb{Z}\}$, if $\hat{\psi}$ has a point of continuity and $\mathcal{WP}(\psi, \mathcal{M})$ is a frame for $L^2(\mathbb{R})$, then Λ is the finite union of logarithmically separated sets. A similar result was shown in the case that $\mathcal{M} = \{(1, x, y) : x \in \Lambda, y \in \mathbb{Z})$ for some subset $\Lambda \subset \mathbb{R}$. In this paper, we will obtain similar results when $\mathcal{M} = \{(x, y, z) : (x, y) \in \mathcal{B}, z \in \mathbb{Z}\}$, where $\mathcal{B}$ is an arbitrary subset of $\mathbb{R}^+ \times \mathbb{R}$. Our results do not seem to generalize to $\mathbb{R}^n$ as readily as the results in [4].

In the next section, we introduce our notations and definitions. In Section 3 we present general results about restrictions on the possible values of dimensions for arbitrary sets. In Section 4 we state and prove our main result (Theorem 20) concerning the necessary conditions for existence of wave packet frames. We also provide large families of new, non-standard examples of wave packet frames.

2 Preliminaries

For $x \in \mathbb{R}^+$ and $y, z \in \mathbb{R}$, let D_x, T_y, and M_z be the unitary operators acting on $L^2(\mathbb{R})$ given by dilations, translations, and modulations, respectively:

$$D_x(f)(t) = \sqrt{x} f(xt), \quad T_y(f)(t) = f(t-y), \quad M_z(f)(t) = e^{2\pi i t z} f(t).$$

With this notation, the definition of a wave packet system given in (1) can be rewritten as $\mathcal{WP}(\psi, \mathcal{M}) = \{D_x T_y M_z \psi : (x,y,z) \in \mathcal{M}\}$.

Motivated by Beurling density for subsets of $\mathbb{R}^n$ and affine density for subsets of the affine group (see [8]), we introduce a notion of density for subsets of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ in the following way. First we observe that we can equip the set $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ with the group multiplication

$$(x,y,z) \cdot (x', y', z') = (xx', x'y + y', \tfrac{z}{x'} + z').$$

Throughout the paper this group, which is sometimes referred to as the *affine Weyl-Heisenberg group*, will be denoted by $\mathbb{G}$. Now let $h > 0$ and let Q_h be the set

$$Q_h = [e^{-h}, e^h] \times [-h, h] \times [-h, h].$$

For any $(x,y,z) \in \mathbb{G}$, we let $Q_h(x,y,z)$ be the set Q_h left-translated via the group action so that it is "centered" at (x,y,z), i.e.,

$$\begin{aligned} Q_h(x,y,z) &= (x,y,z) \cdot Q_h \\ &= \Big\{\Big(xx', x'y + y', \tfrac{z}{x'} + z'\Big) : x' \in [e^{-h}, e^h], y' \in [-h,h], z' \in [-h,h]\Big\}. \end{aligned}$$

The left-invariant Haar measure μ on $\mathbb{G}$ is $\frac{dx}{x}\,dy\,dz$, and thus we have

$$\mu(Q_h(x,y,z)) = \mu(Q_h) = \int_{e^{-h}}^{e^h} \int_{-h}^{h} \int_{-h}^{h} \frac{dx}{x}\,dy\,dz = 8h^3.$$

Let $\mathcal{M}$ be a discrete subset of $\mathbb{G}$, and let $A > 0$. Then the *lower Beurling density of $\mathcal{M}$ (with respect to A)* is defined by

$$\mathcal{D}_A^-(\mathcal{M}) = \liminf_{h \to \infty} \inf_{(x,y,z) \in \mathbb{G}} \frac{\#(\mathcal{M} \cap Q_h(x,y,z))}{h^A},$$

and the *upper Beurling density of $\mathcal{M}$ (with respect to A)* is defined by

$$\mathcal{D}_A^+(\mathcal{M}) = \limsup_{h \to \infty} \sup_{(x,y,z) \in \mathbb{G}} \frac{\#(\mathcal{M} \cap Q_h(x,y,z))}{h^A}.$$

The introduction of the parameter A is justified by the observation that for $\mathcal{M} = \{1\} \times a\mathbb{Z} \times b\mathbb{Z}$, $a, b > 0$, which is the set of parameters of (the Fourier transform of) an arbitrary regular Gabor system $\{M_{am}T_{bn}g : m, n \in \mathbb{Z}\}$, $g \in L^2(\mathbb{R})$, it can be easily checked that $\mathcal{D}_3^+(\mathcal{M}) = 0$. Thus, using the Haar measure of the boxes $Q_h(x, y, z)$, as it is done in the definition of Beurling density and affine density, would lead to a notion which is useless for the most important Gabor systems. However, $\mathcal{D}_2^+(\mathcal{M}) = (ab)^{-1}$, which corresponds to the usual Beurling density in the plane. In particular, the parameter A allows us to find densities of sets which are embedded in larger dimensional spaces.

Now, motivated by the definition of the *mass dimension* of a discrete set (see, for example [2,16]), we define the *lower Beurling dimension of* $\mathcal{M} \subset \mathbb{G}$ to be

$$\dim^-(\mathcal{M}) = \inf\{A > 0 : \mathcal{D}_A^-(\mathcal{M}) < \infty\},$$

and the *upper Beurling dimension of* $\mathcal{M} \subset \mathbb{G}$ to be

$$\dim^+(\mathcal{M}) = \sup\{A > 0 : \mathcal{D}_A^+(\mathcal{M}) > 0\}.$$

It immediately follows that $\dim^-(\mathcal{M}) \leq \dim^+(\mathcal{M})$ for all $\mathcal{M} \subset \mathbb{G}$. We remark that in what follows, we will refer to the lower and upper Beurling dimensions as lower and upper dimensions.

Notation. Throughout this paper let $\mathbf{1}_U$ denote the characteristic function of a Lebesgue measurable set $U \subset \mathbb{R}^d$, and let $|U|$ denote its Lebesgue measure.

3 General results

In this section we study the properties of the upper and lower (Beurling) dimensions of arbitrary subsets of $\mathbb{G}$.

First we give a useful reinterpretation of finite upper density when $A = 3$.

Proposition 1 *Let $\mathcal{M}$ be a discrete subset of $\mathbb{G}$.*

(1) The following conditions are equivalent.
(a) $\mathcal{D}_3^+(\mathcal{M}) < \infty$.
(b) There exists some $h > 0$ such that

$$\sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x, y, z)) < \infty.$$

(2) Also the following conditions are equivalent.
(a) $\mathcal{D}_3^-(\mathcal{M}) > 0$.
(b) There exists some $h > 0$ such that

$$\inf_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x, y, z)) > 0.$$

PROOF. The implication (a) $\Rightarrow$ (b) follows immediately in both cases.

To prove (b) $\Rightarrow$ (a) for part (1), suppose there exists $h > 0$ such that $R = \sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x,y,z)) < \infty$. Let $r > 1$. It is easy to compute that Q_{rh} is contained in

$$\bigcup_{i=1}^{r} \bigcup_{j,k=0}^{\lceil e^{2h}(r-1)\rceil} Q_h(e^{h(2i-1-r)}, e^h h(1-r) + 2je^{-h}h, e^h h(1-r) + 2ke^{-h}h).$$

Hence, for each $(x,y,z) \in \mathbb{G}$, also $Q_{rh}(x,y,z) = (x,y,z)\cdot Q_{rh}$ is covered by $r\lceil e^{2h}(r-1)\rceil^2$ disjoint sets $Q_h(x_l, y_l, z_l)$ with $(x_l, y_l, z_l) \in \mathbb{G}$ chosen appropriately. This implies

$$\begin{aligned}\sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_{rh}(x,y,z)) &\le r\lceil e^{2h}(r-1)\rceil^2 \sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x,y,z)) \\ &\le r\lceil e^{2h}(r-1)\rceil^2 R < \infty.\end{aligned}$$

Thus

$$\mathcal{D}_3^+(\mathcal{M}) \le \limsup_{r\to\infty} \frac{r\lceil e^{2h}(r-1)\rceil^2 R}{r^3 h^3} = \frac{e^{4h} R}{h^3} < \infty.$$

The argument that (b) $\Rightarrow$ (a) for part (2) is very similar to part (1), where here we use the fact that the disjoint unions

$$\bigcup_{i=1}^{r} \bigcup_{j,k=1}^{\lfloor e^{-2h}(r-1)\rfloor} Q_h(e^{h(2i-1-r)}, e^{-h}h(1-r) + 2je^{h}h, e^{-h}h(1-r) + 2ke^{h}h)$$

are contained in Q_{rh}, and so we omit the details. $\square$

Theorem 2 *Let $\mathcal{M}$ be a subset of $\mathbb{G}$. Then,*

(1) $\dim^+(\mathcal{M}) \in [0,3] \cup \{\infty\}$*, and*
(2) $\dim^-(\mathcal{M}) \in \{0\} \cup [3,\infty]$.

PROOF. Assume that we have $\dim^+(\mathcal{M}) =: d > 3$ and $d < \infty$. This implies $\mathcal{D}_3^+(\mathcal{M}) = \infty$. By Proposition 1 (1), $\sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x,y,z)) = \infty$ for all $h > 0$. Hence $\mathcal{D}_{d+1}^+(\mathcal{M}) = \infty$, a contradiction to $d = \sup\{A : \mathcal{D}_A^+(\mathcal{M}) > 0\}$. This proves (1).

To show (2) assume that $\dim^-(\mathcal{M}) =: d \in (0,3)$. This implies $\mathcal{D}_3^-(\mathcal{M}) = 0$. By Proposition 1 (2), $\inf_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x,y)) = 0$ for all $h > 0$. Hence also $\mathcal{D}_{\frac{d}{2}}^-(\mathcal{M}) = 0$, which contradicts $d = \inf\{A : \mathcal{D}_A^-(\mathcal{M}) < \infty\}$. $\square$

Theorem 3 *Let $\psi \in L^2(\mathbb{R})$ and $\mathcal{M}$ be a discrete subset of $\mathbb{G}$. If $\mathcal{WP}(\psi,\mathcal{M})$ is a Bessel system, then $\mathcal{D}_A^+(\mathcal{M}) < \infty$ for all $A \ge 3$.*

PROOF. Let us remark that the following proof will use ideas of the proof of [8, Theorem 1(a)].

In the following we let π, which maps $\mathbb{G}$ into the unitary operators on $L^2(\mathbb{R})$, be defined by $\pi(x,y,z)f(t) = D_x T_y M_z f(t)$. Let $\psi \in L^2(\mathbb{R})$ and let $\mathcal{M} \subset \mathbb{G}$ be such that $\mathcal{D}_3^+(\mathcal{M}) = \infty$. Fix some $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$. Since $(x,y,z) \mapsto \langle f, \pi(x,y,z)\psi\rangle$ is continuous and non-zero, there exists $(r,s,t) \in \mathbb{G}$ and $h > 0$ with

$$\delta = \inf_{(x,y,z)\in Q_h(r,s,t)} |\langle f, \pi(x,y,z)\psi\rangle| > 0.$$

Now choose any $N > 0$. Since $\mathcal{D}_3^+(\mathcal{M}) = \infty$, Proposition 1 (1) implies that

$$\sup_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M} \cap Q_h(x,y,z)) = \infty.$$

Thus there exists $(o,p,q) \in \mathbb{G}$ with $\#(\mathcal{M} \cap Q_h(o,p,q)) > N$. Defining $g \in L^2(\mathbb{R})$ by

$$g = \pi((o,p,q)\cdot(r,s,t)^{-1})f$$

and using the fact that

$$(x,y,z) \in Q_h(o,p,q) \Longrightarrow (r,s,t)\cdot(o,p,q)^{-1}\cdot(x,y,z) \in Q_h(r,s,t),$$

we obtain

$$\begin{aligned}
&\sum_{(x,y,z)\in\mathcal{M}} |\langle g, \pi(x,y,z)\psi\rangle|^2 \\
&\geq \sum_{(x,y,z)\in\mathcal{M}\cap Q_h(o,p,q)} \left|\left\langle \pi((o,p,q)\cdot(r,s,t)^{-1})f, \pi(x,y,z)\psi\right\rangle\right|^2 \\
&= \sum_{(x,y,z)\in\mathcal{M}\cap Q_h(o,p,q)} \left|\left\langle f, \pi((r,s,t)\cdot(o,p,q)^{-1}\cdot(x,y,z))\psi\right\rangle\right|^2 > N\delta^2.
\end{aligned}$$

Thus $\mathcal{WP}(\psi,\mathcal{M})$ does not possess a finite upper frame bound, i.e., it is not a Bessel system. Since $\mathcal{D}_3^+(\mathcal{M}) < \infty$ implies $\mathcal{D}_A^+(\mathcal{M}) < \infty$ for all $A \geq 3$, the proof is complete. $\square$

In other words, if $\mathcal{WP}(\psi,\mathcal{M})$ is a Bessel system, then zero and three are the only two possible values for $\dim^-(\mathcal{M})$. Wavelet frames and Gabor frames are examples of wave packet frames that satisfy the condition $\dim^-(\mathcal{M}) = 0$. For some non-standard examples, we refer the reader to the next section. It is still unknown to the authors if there can exist a general wave packet frame $\mathcal{WP}(\psi,\mathcal{M})$ for which $\dim^-(\mathcal{M}) = 3$. However, we can answer this question when the sets of parameters of wave packet systems have a special form: i.e., $\mathcal{M} = \mathcal{B} \times \mathbb{Z}$, where $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$. Wave packet systems with such sets of parameters have been recently studied in [9–12].

We end this section by reinterpreting the definition of dimension. This result will be used in the next section.

Proposition 4 *Let $\mathcal{M}$ be a subset of $\mathbb{G}$. Then,*

(1) $\dim^-(\mathcal{M}) = \sup\{A > 0 : \mathcal{D}_A^-(\mathcal{M}) > 0\}$, *and*
(2) $\dim^+(\mathcal{M}) = \inf\{A > 0 : \mathcal{D}_A^+(\mathcal{M}) < \infty\}$.

PROOF. We provide here only the proof of the first claim. The proof of the second equality follows in a similar way. Also, for the sake of brevity, throughout this proof we shall use ν_h^- to denote $\inf_{(x,y,z)\in\mathbb{G}} \#(\mathcal{M}\cap Q_h(x,y,z))$.

First, we note that if there exists $0 < A_0 < \infty$ such that for all $\delta > 0$ we have either

$$\liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0}} = 0 \quad \text{and} \quad \liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0-\delta}} = \infty$$

or

$$\liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0}} = \infty \quad \text{and} \quad \liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0+\delta}} = 0,$$

then our claim follows immediately.

Hence, we only need to consider the case where there exists $0 < A_0 < \infty$ such that

$$0 < a := \liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0}} < \infty. \tag{3}$$

In such case, in order to finish the proof, it suffices to show that for each $\delta > 0$ we have

$$\liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0+\delta}} = 0 \quad \text{and} \quad \liminf_{h\to\infty} \frac{\nu_h^-}{h^{A_0-\delta}} = \infty. \tag{4}$$

By hypothesis (3), there exists a subsequence $(h_n)_n$ with $\lim_{n\to\infty} \frac{\nu_{h_n}^-}{h_n^{A_0}} = a$. Now let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $|\frac{\nu_{h_n}^-}{h_n^{A_0}} - a| < \epsilon$ for all $n \geq N$. This yields

$$\left| \frac{\nu_{h_n}^-}{h_n^{A_0+\delta}} - \frac{a}{h_n^\delta} \right| < \frac{\epsilon}{h_n^\delta} \quad \forall\ n \geq N.$$

Without loss of generality we may assume that $\frac{\epsilon}{h_n^\delta} < \epsilon$ for all $n \geq N$. Moreover, we have $\lim_{n\to\infty} \frac{a}{h_n^\delta} = 0$. This shows the first claim in (4).

For the proof of the second claim, let $M > 0$ be arbitrary. For each $\epsilon > 0$, the set $\{h > 0 : \frac{\nu_h^-}{h^{A_0}} < a-\epsilon\}$ is finite. Hence also the set $\{h > 0 : \frac{\nu_h^-}{h^{A_0-\delta}} < (a-\epsilon)h^\delta\}$ is finite. Fix some $\epsilon > 0$ and let $h_0 > 0$ be defined by $(a-\epsilon)h_0^\delta = M$. Then we have

$$\{h > h_0 : \frac{\nu_h^-}{h^{A_0-\delta}} < M\} \subseteq \{h > 0 : \frac{\nu_h^-}{h^{A_0-\delta}} < (a-\epsilon)h^\delta\}.$$

This implies that the set $\{h > h_0 : \frac{\nu_h^-}{h^{A_0-\delta}} < M\}$ is finite. Since M was chosen arbitrarily, this proves the second claim in (4). □

4 Case of integer modulations

In this section, we restrict ourselves to the simpler situation where $\mathcal{M} \subset \mathbb{G}$ has the form $\mathcal{M} = \mathcal{B} \times \mathbb{Z}$, where $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$. This situation is still more general for the types of problems we are considering than the most general situation considered so far in [4].

First we introduce the notion of upper and lower dimension for subsets of $\mathbb{R}^+ \times \mathbb{R}$. In Lemma 8, we will then establish a relationship between the dimension of $\mathcal{B} \times \mathbb{Z}$ and the dimension of $\mathcal{B}$, when $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$.

Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the *affine group* with multiplication given by

$$(x, y) \cdot (x', y') = (xx', x'y + y').$$

Further, let $h > 0$ and let Q_h be the set $Q_h = [e^{-h}, e^h] \times [-h, h]$. For any $(x, y) \in \mathbb{A}$ we define $Q_h(x, y) = (x, y) \cdot Q_h$.

Now let $\mathcal{B}$ be a discrete subset of $\mathbb{A}$, and let $A > 0$. Then the *upper (Beurling) density of* $\mathcal{B}$ *(with respect to* A*)* is defined by

$$\mathcal{D}_A^+(\mathcal{B}) = \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\#(\mathcal{B} \cap Q_h(x, y))}{h^A}$$

and the *lower (Beurling) density of* $\mathcal{B}$ *(with respect to* A*)* is defined similarly. Based on this definition the lower and upper dimensions of $\mathcal{B} \subset \mathbb{A}$ are defined in the same fashion as the corresponding dimensions of $\mathcal{M} \subset \mathbb{G}$.

Remark 5 *Note that, if* $\mathcal{B} \subset \mathbb{A}$*, and we define* $\mathcal{M} = \mathcal{B} \times \{0\} \subset \mathbb{G}$*, then the lower dimension of* $\mathcal{B}$ *and* $\mathcal{M}$ *are not in general the same. Indeed, in this case,* $\dim^-(\mathcal{M}) = 0$*, so one must be careful to specify which group one is considering* $\mathcal{B}$ *to be contained in. To make the context clear, we will reserve* $\mathcal{B}$ *for subsets of* $\mathbb{A}$*, and* $\mathcal{M}$ *for subsets of* $\mathbb{G}$*.*

The following proposition and its proof are similar to Proposition 1.

Proposition 6 *Let* $\mathcal{B}$ *be a discrete subset of* $\mathbb{A}$*.*

(1) The following conditions are equivalent.
(a) $\mathcal{D}_2^+(\mathcal{B}) < \infty$*.*

(b) There exists some $h > 0$ such that

$$\sup_{(x,y)\in\mathbb{A}} \#(\mathcal{B} \cap Q_h(x,y)) < \infty.$$

(2) Also the following conditions are equivalent.
(a) $\mathcal{D}_2^-(\mathcal{B}) > 0$.
(b) There exists some $h > 0$ such that

$$\inf_{(x,y)\in\mathbb{A}} \#(\mathcal{B} \cap Q_h(x,y)) > 0.$$

Corollary 7 *If $\mathcal{B} \subset \mathbb{A}$, then $\mathcal{D}_2^-(\mathcal{B}) = 0$ implies $\mathcal{D}_A^-(\mathcal{B}) = 0$ for all $0 < A < \infty$.*

PROOF. By Proposition 6 (2), $\mathcal{D}_2^-(\mathcal{B}) = 0$ implies that, for all $h > 0$, we have $\inf_{(x,y)\in\mathbb{A}} \#(\mathcal{B} \cap Q_h(x,y)) = 0$. This immediately proves $\mathcal{D}_A^-(\mathcal{B}) = 0$ for all $0 < A < \infty$. □

Next, we show how the dimension of $\mathcal{B} \subset \mathbb{A}$ is related to the dimension of $\mathcal{B} \times \mathbb{Z} \subset \mathbb{G}$. Note that the notation $\mathcal{D}$ can refer to either the density of a subset of $\mathbb{A}$ or a subset of $\mathbb{G}$, depending on the context.

Lemma 8 *Let $\mathcal{B}$ be a discrete subset of $\mathbb{A}$. Then, for each $1 < A < \infty$, we have*

$$\mathcal{D}_A^-(\mathcal{B} \times \mathbb{Z}) = 2\mathcal{D}_{A-1}^-(\mathcal{B}) \quad \text{and} \quad \mathcal{D}_A^+(\mathcal{B} \times \mathbb{Z}) = 2\mathcal{D}_{A-1}^+(\mathcal{B}).$$

PROOF. Fix $h > 0$ and $(x,y,z) \in \mathbb{G}$. For $(r,s,t) \in \mathcal{B} \times \mathbb{Z}$, we have $(r,s,t) \in Q_h(x,y,z)$ if and only if

$$(\tfrac{r}{x}, s - \tfrac{ry}{x}, t - \tfrac{xz}{r}) = (x,y,z)^{-1} \cdot (r,s,t) \in [e^{-h}, e^h] \times [-h,h] \times [-h,h].$$

Hence, there are at least $2h$ and at most $2h+1$ integers t satisfying this condition. Moreover,

$$(\tfrac{r}{x}, s - \tfrac{ry}{x}) \in [e^{-h}, e^h] \times [-h,h] \iff (r,s) \in Q_h(x,y).$$

This shows

$$2h\#(\mathcal{B} \cap Q_h(x,y)) \le \#((\mathcal{B} \times \mathbb{Z}) \cap Q_h(x,y,z)) \le (2h+1)\#(\mathcal{B} \cap Q_h(x,y)).$$

Thus $2\mathcal{D}_{A-1}^-(\mathcal{B}) \le \mathcal{D}_A^-(\mathcal{B} \times \mathbb{Z}) \le 2\mathcal{D}_{A-1}^-(\mathcal{B})$, and similarly $\mathcal{D}_A^+(\mathcal{B} \times \mathbb{Z}) = 2\mathcal{D}_{A-1}^+(\mathcal{B})$. □

The factor of 2 that appears in Lemma 8 is a consequence of choosing in our considerations intervals of length $2h$ in the definition of Q_h.

The following result shows that not all dimensions can be attained. Compare to Theorem 2.

Theorem 9 *Let $\mathcal{B}$ be a discrete subset of $\mathbb{A}$. Then*

(1) $\dim^+(\mathcal{B}) \in [0,2] \cup \{\infty\}$, *and*
(2) $\dim^-(\mathcal{B}) \in \{0\} \cup [2,\infty]$.

PROOF. We will only prove part (1). Part (2) follows in a similar way.

By Lemma 8, we have

$$\dim^+(\mathcal{B}) = \sup\{A > 0 : \tfrac{1}{2}\mathcal{D}^+_{A+1}(\mathcal{B} \times \mathbb{Z}) > 0\} = \dim^+(\mathcal{B} \times \mathbb{Z}) - 1.$$

Applying Theorem 2 yields the claim. □

Now, we turn to relating the existence of frames to the dimension of $\mathcal{B}$. A first necessary condition on the dimension is given by the following theorem.

Theorem 10 *Let $\mathcal{B}$ be a discrete subset of $\mathbb{A}$. If there exists a non-zero function $\psi \in L^2(\mathbb{R})$ such that $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ has an upper frame bound, then $\dim^-(\mathcal{B}) = 0$.*

Lemma 11 *Suppose $\mathcal{B} \subset \mathbb{A}$ has the following property: for all $A \subset \mathbb{R}$ with positive measure and all $n \in \mathbb{N}$, there exist $(x_1, y_1), \ldots, (x_n, y_n) \in \mathcal{B}$ such that*

$$\left| \bigcap_{i=1}^{n} \frac{1}{x_i}\Big(A + y_i\Big) \right| > 0.$$

Then, for every non-zero $\psi \in L^2(\mathbb{R})$, $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ fails to be a Bessel system.

PROOF of Lemma 11. Let $\psi \in L^2(\mathbb{R})$, $\psi \neq 0$. Then there exists a set $A \subset \mathbb{R}$ of positive measure such that $|\psi(x)| \geq C > 0$ for almost all $x \in A$. By reducing to a subset, we may assume that there exists a constant $K > 0$ such that, for every function $f \in L^2(\mathbb{R})$ with support in A, we have

$$\sum_{z \in \mathbb{Z}} |\langle f, M_z \psi \rangle|^2 \geq K \|f\|^2.$$

Since the operators D_x and T_y are unitary, for every $(x, y) \in \mathcal{B}$ and for all functions $f \in L^2(\mathbb{R})$ with support in $\frac{1}{x}(A + y)$, we obtain

$$\sum_{z \in \mathbb{Z}} |\langle f, D_x T_y M_z \psi \rangle|^2 \geq K \|f\|^2. \qquad (5)$$

By hypothesis, for any $n \in \mathbb{N}$, we can choose $(x_1, y_1), \dots, (x_n, y_n) \in \mathcal{B}$ such that the set $U = \bigcap_{i=1}^n \frac{1}{x_i}(A + y_i)$ has positive measure. Using (5), this implies

$$\begin{aligned}\sum_{(x,y)\in\mathcal{B},z\in\mathbb{Z}} |\langle \mathbf{1}_U, D_x T_y M_z \psi\rangle|^2 &\geq \sum_{i=1}^n \sum_{z\in\mathbb{Z}} |\langle \mathbf{1}_U, D_{x_i} T_{y_i} M_z \psi\rangle|^2 \\ &\geq \sum_{i=1}^n K\|\mathbf{1}_U\|^2 = nK\|\mathbf{1}_U\|^2.\end{aligned}$$

Thus, there exists no finite upper frame bound, since n is arbitrary. □

The proof of the following lemma, which is just a version of Bonferroni's inequality, is obtained by induction on k and the inequality $|B| \geq \sum_{i=1}^N |A_i| - \sum_{i_1<i_2} |A_{i_1} \cap A_{i_2}|$.

Lemma 12 *If $\{A_i\}_{i=1}^N$ are measurable subsets of the measurable set B and k is a positive integer such that $\sum_{i=1}^N |A_i| > k|B|$, then there exist $1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq N$ such that*

$$\left|\bigcap_{j=1}^k A_{i_j}\right| > 0.$$

PROOF of Theorem 10. To prove our claim we argue by contradiction and assume that $\mathcal{B}$ is a discrete subset of $\mathbb{A}$ with $\dim^-(\mathcal{B}) > 0$. In order to apply Lemma 11, let A be a set of positive measure. Without loss of generality we can assume that it is contained in some interval $[a, b]$. Further we assume that $a > 0$. The other cases can be dealt with in a similar way. Let $x > 0$ and note that $\frac{1}{x}(A + y) \subset [0, 1]$ whenever $y \geq -a$ and $y \leq x - b$. Therefore, in view of Lemmas 11 and 12, and since $|\frac{1}{x}(A + y)| = \frac{1}{x}$ for all $x > 0$, $y \in \mathbb{R}$, it suffices to show that

$$\sum_{\{(x,y)\in\mathcal{B}: y\geq -a, y\leq x-b\}} \tfrac{1}{x} = \infty. \tag{6}$$

To prove this, let $R = \{(x, y) \in \mathbb{A} : y \geq -a,\ y \leq x-b\}$. Since $\dim^-(\mathcal{B}) > 0$, by Theorem 9, it follows that $\dim^-(\mathcal{B}) \geq 2$. Applying Lemma 8 and Proposition 4 (1) now yields $\sup\{A > 0 : \mathcal{D}_A^-(\mathcal{B}\times\mathbb{Z}) > 0\} \geq 3$. Thus $\mathcal{D}_A^-(\mathcal{B}\times\mathbb{Z}) > 0$ for all $0 \leq A < 3$. Using Lemma 8 again, this implies $\mathcal{D}_A^-(\mathcal{B}) > 0$ for all $0 \leq A < 2$. Consequently, there exists some $h > 0$ such that $\inf\{\#(Q_h(x, y)\cap\mathcal{B}) : (x, y) \in \mathbb{A}\} \geq 1$. Fix $x_0 > 0$, and define $(x_n, y_k) \in \mathbb{A}$, $k, n \geq 0$ by

$$x_n = e^{2nh} x_0 \quad \text{and} \quad y_k = (2k + 1)he^h.$$

It is straightforward to check that the sets $Q_h(x_n, y_k)$, where $n \geq 0$, $0 \leq k \leq \frac{x_n e^{-h} - b}{2h} - \frac{3}{2}$ are pairwise disjoint subsets of R. By choice of h, each of these sets contains at least one element of $\mathcal{B}$. In particular, for each $n \geq 0$ and $0 \leq k \leq \frac{x_n e^{-h} - b}{2h} - \frac{3}{2}$, there exist distinct elements $(z_n, w_k) \in \mathcal{B}$ such that for all $n \geq 0$, $0 < z_n \leq e^{(2n+1)h} x_0$. If we let $K_n := \lfloor \frac{x_n e^{-h} - b}{2h} - \frac{3}{2} \rfloor$, we obtain

$$\sum_{(x,y)\in\mathcal{B}\cap R} \frac{1}{x} \geq \sum_{n=0}^{\infty} \frac{K_n}{e^{(2n+1)h} x_0}.$$

Since

$$\lim_{n\to\infty} \frac{K_n}{e^{(2n+1)h} x_0} = \frac{1}{2he^{2h}} > 0,$$

(6) is established, which finishes the proof. □

Now, we wish to construct examples of wave packet frames with specified dimension. A useful tool is the following proposition.

Proposition 13 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 such that $\frac{x_n}{x_{n+1}} \geq K > 1$ for some constant K. Then, there exist a real sequence $(y_n)_{n\in\mathbb{N}}$ and some constant $M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$ and*

$$\mathcal{WP}(\mathbf{1}_{[0,1]}, \{(x_n, y_n) : n \in \mathbb{N}\} \times \mathbb{Z})$$

is an orthonormal basis for $L^2(\mathbb{R})$.

PROOF. It suffices to prove the existence of a real bounded sequence $(y_n)_{n\in\mathbb{N}}$ such that if we define $E_n = \frac{1}{x_n}([0,1] + y_n)$, then $\{E_n : n \in \mathbb{N}\}$ is a measurable tiling of $\mathbb{R}$. In the following we will construct a sequence $(y_n)_{n\in\mathbb{N}}$ so that $\{E_n : n \in \mathbb{N}\}$ tiles $\mathbb{R}^+$. Then $\mathbb{R}^-$ can be dealt with in a similar way.

First we choose $y_1 = 0$ so that we obtain $E_1 = [0, \frac{1}{x_1}]$. Now we define the sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \sum_{i=1}^{n-1} \frac{x_n}{x_i}, \quad \forall\, n > 1.$$

To prove the boundedness of the sequence $(y_n)_{n\in\mathbb{N}}$, for each $n \in \mathbb{N}$, we compute

$$\begin{aligned} y_n &= \sum_{i=1}^{n-1} \frac{x_n}{x_i} \\ &= \frac{x_n}{x_{n-1}} + \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} + \ldots + \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} \frac{x_{n-2}}{x_{n-3}} \cdot \ldots \cdot \frac{x_2}{x_1} \\ &\leq K^{-1} + K^{-2} + \ldots + K^{-n+1} \\ &\leq \sum_{i=1}^{\infty} K^{-i} < \infty. \end{aligned}$$

As defined above we have $E_n = [\frac{y_n}{x_n}, \frac{y_n}{x_n} + \frac{1}{x_n}]$. Since, by definition of $(y_n)_{n\in\mathbb{N}}$,

$$\frac{y_n}{x_n} + \frac{1}{x_n} = \sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n} = \frac{y_{n+1}}{x_{n+1}},$$

it follows that the set $\{E_n : n \in \mathbb{N}\}$ tiles $\mathbb{R}^+$. $\square$

Theorem 14 *Let $\psi = \mathbf{1}_{[0,1]}$. For every $0 < a \leq 1$, there exists a discrete subset $\mathcal{B} \subset \mathbb{A}$ such that $\dim^+(\mathcal{B}) = a$ and $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$.*

PROOF. For $a = 1$, one can choose the Gabor system, i.e., $\mathcal{B} = \{1\} \times \mathbb{Z}$.

Now suppose that $0 < a < 1$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = e^{-2n^{\frac{1}{a}}}$. It is easy to check that this sequence satisfies the conditions of Proposition 13. Thus there exists a sequence $(y_n)_{n\in\mathbb{N}}$, which is bounded by some constant $M > 0$, such that $\mathcal{WP}(\psi, \{(x_n, y_n) : n \in \mathbb{N}\} \times \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$. Let $\mathcal{B} = \{(x_n, y_n) : n \in \mathbb{N}\}$. By inspection, for all $k \in \mathbb{N}$ such that $k^{\frac{1}{a}} > M$, we obtain

$$\begin{aligned}\sup_{(x,y)\in\mathbb{A}} \#(Q_{k^{\frac{1}{a}}}(x,y) \cap \mathcal{B}) &\geq \#(Q_{k^{\frac{1}{a}}}(e^{-k^{\frac{1}{a}}}, 0) \cap \mathcal{B}) \\ &\geq \#([e^{-2k^{\frac{1}{a}}}, 1] \cap \{x_n : n \in \mathbb{N}\}) = k.\end{aligned}$$

Therefore,

$$\limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\#(Q_h(x,y) \cap \mathcal{B})}{h^a} \geq \limsup_{k\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\#(Q_{k^{\frac{1}{a}}}(x,y) \cap \mathcal{B})}{k} > 0. \quad (7)$$

It remains to show that the first term in (7) is also finite. To prove this, we first compute an upper bound for $\sup_{x\in\mathbb{R}^+} \#([xe^{-h}, xe^h] \cap \mathcal{B}_1)$, where $h > 0$, $\mathcal{B}_1 = \{x_n : n \in \mathbb{N}\}$ and $k \in \mathbb{N}$ is chosen in such a way that $k^{\frac{1}{a}} < h \leq (k+1)^{\frac{1}{a}}$. We obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^+} \#([xe^{-h}, xe^h] \cap \mathcal{B}_1) \\ &\leq \sup_{x\in\mathbb{R}^+} \#([xe^{-(k+1)^{\frac{1}{a}}}, xe^{(k+1)^{\frac{1}{a}}}] \cap \mathcal{B}_1) \\ &\leq \sup_{m\in\mathbb{Z}} \#([e^{-2m^{\frac{1}{a}}-2(k+1)^{\frac{1}{a}}}, e^{-2m^{\frac{1}{a}}}] \cap \mathcal{B}_1) \\ &= \sup_{m\in\mathbb{Z}} \#(\{j \in \mathbb{Z} : e^{-2m^{\frac{1}{a}}-2(k+1)^{\frac{1}{a}}} \leq e^{-2(m+j)^{\frac{1}{a}}} \leq e^{-2m^{\frac{1}{a}}}\}) \\ &= \sup_{m\in\mathbb{Z}} \#(\{j \geq 0 : -2(m+j)^{\frac{1}{a}} \geq -2m^{\frac{1}{a}} - 2(k+1)^{\frac{1}{a}}\}) \\ &= \sup_{m\in\mathbb{Z}} \#(\{j \geq 0 : j \leq (m^{\frac{1}{a}} + (k+1)^{\frac{1}{a}})^a - m\}) \leq k+2,\end{aligned}$$

where the second inequality is due to the fact that we can move the right hand endpoint to the left so that it touches a point in $\mathcal{B}_1$, and the last inequality is because $\frac{1}{a} \geq 1$ and $(m^{1/a} + (k+1)^{1/a})^a \leq m + k + 1$ (the triangle inequality in $\ell^2_{\frac{1}{a}}$). This yields

$$\sup_{x \in \mathbb{R}^+} \frac{\#([xe^{-h}, xe^h] \cap \mathcal{B}_1)}{h^a} \leq \frac{k+2}{k}. \tag{8}$$

Since $Q_h(x,y) \subset [xe^{-h}, xe^h] \times \mathbb{R}$ and $\#(\mathcal{B}_1 \cap (\{z\} \times \mathbb{R})) \leq 1$ for all $z \in \mathbb{R}^+$, we obtain

$$\#(Q_h(x,y) \cap \mathcal{B}) \leq \#([xe^{-h}, xe^h] \cap \mathcal{B}_1). \tag{9}$$

By (8) and (9), it follows that

$$\limsup_{h \to \infty} \sup_{(x,y) \in \mathbb{A}} \frac{\#(Q_h(x,y) \cap \mathcal{B})}{h^a} \leq \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^+} \frac{\#([xe^{-h}, xe^h] \cap \mathcal{B}_1)}{h^a} < \infty. \tag{10}$$

Finally, combining (7) and (10) yields $\dim^+(\mathcal{B}) = a$. $\square$

The most difficult examples to construct are for large dimensions. Thus, let $1 < a \leq 2$ be fixed. Then, for each pair (w,k) with $w > 0$ and $k > \frac{1}{2}$, we define

$$x_n = x_n(w,k) = we^{-k} \left(\frac{k + \frac{1}{2}}{k - \frac{1}{2}} \right)^{n^{\frac{2}{a}}}, \quad n \geq 0.$$

Further, for each $k \in \mathbb{N}$, we choose $w_k > 0$ and $N_k > 0$ as

$$w_k = e^{e^k + k} \quad \text{and} \quad N_k = \frac{2k}{\ln\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)}.$$

Now, let the subset $\mathcal{B}_0 \subset \mathbb{A}$ be defined (for $K > 0$ to be chosen later) by

$$\mathcal{B}_0 = \mathcal{B}_0(a) = \left\{ (x_n(w_k,k), k) : k \geq K,\ 0 \leq n^{\frac{2}{a}} \leq N_k,\ k, n \in \mathbb{N} \right\}. \tag{11}$$

Lemma 15 *Let $1 < a \leq 2$ and $\psi = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$. Then $\mathcal{WP}(\psi, \mathcal{B}_0 \times \mathbb{Z})$ is a Bessel system for $L^2(\mathbb{R})$ (with Bessel constant 1).*

PROOF. It suffices to show that the set

$$\left\{ \tfrac{1}{x}([-\tfrac{1}{2}, \tfrac{1}{2}] + y) : (x,y) \in \mathcal{B}_0 \right\} \tag{12}$$

is a pairwise disjoint collection of subsets of $\mathbb{R}$. For this, we observe that

$$\frac{1}{x_n(w_k,k)}([-\tfrac{1}{2}, \tfrac{1}{2}] + k) = e^{-e^k} \left[\frac{(k - \frac{1}{2})^{n^{\frac{2}{a}}+1}}{(k+\frac{1}{2})^{n^{\frac{2}{a}}}}, \frac{(k-\frac{1}{2})^{n^{\frac{2}{a}}}}{(k+\frac{1}{2})^{n^{\frac{2}{a}}-1}} \right]. \tag{13}$$

First we consider the case when $k \in \mathbb{N}$ is fixed. Then it is sufficient to prove that

$$\frac{(k-\frac{1}{2})^{(n+1)^{\frac{2}{a}}}}{(k+\frac{1}{2})^{(n+1)^{\frac{2}{a}}-1}} \leq \frac{(k-\frac{1}{2})^{n^{\frac{2}{a}}+1}}{(k+\frac{1}{2})^{n^{\frac{2}{a}}}},$$

which is equivalent to

$$(k+\tfrac{1}{2})^{n^{\frac{2}{a}}-(n+1)^{\frac{2}{a}}+1} \leq (k-\tfrac{1}{2})^{n^{\frac{2}{a}}-(n+1)^{\frac{2}{a}}+1}.$$

This in turn follows immediately from $1 < a \leq 2$ and $n^{\frac{2}{a}} - (n+1)^{\frac{2}{a}} + 1 < 0$.

In order to deal with the general situation when k varies, for each $k \in \mathbb{N}$, we define a subset $A_k \subset \mathbb{R}$ by

$$A_k = \bigcup_{0 \leq n^{\frac{2}{a}} \leq N_k} \frac{1}{x_n(w_k, k)}[-\tfrac{1}{2}+k, \tfrac{1}{2}+k].$$

Using (13), it is easy to check that

$$A_k \subset \left[\frac{(k-\frac{1}{2})^{N_k+1}}{(k+\frac{1}{2})^{N_k} e^{e^k}}, \frac{k+\frac{1}{2}}{e^{e^k}}\right].$$

Therefore, in order to see that the collection of sets in (12) is disjoint, it suffices to show that

$$\frac{k+\frac{3}{2}}{e^{e^{k+1}}} < \frac{(k-\frac{1}{2})^{N_k+1}}{(k+\frac{1}{2})^{N_k} e^{e^k}}. \tag{14}$$

To prove this, note that for large k, the sequence $(N_k)_{k\in\mathbb{N}}$ behaves like $(k^2)_{k\in\mathbb{N}}$ — in fact, it is not difficult to verify that

$$N_k \leq 2k(k+1/2). \tag{15}$$

Hence, for k large enough, we obtain

$$(k+\tfrac{3}{2})\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)^{N_k} \leq ke^{k^2} < e^{e^k} \leq e^{e^{k+1}-e^k}.$$

This implies that there exists some $K > 0$ such that equation (14) is satisfied for all $k \geq K$. Such a K shall be used for the definition of the set $\mathcal{B}_0$ (see (11)). □

The next result is a technical lemma which will be needed in the proof of Theorem 17.

Lemma 16 *Let $\mathcal{B}_k$ denote the intersection of $\mathcal{B}_0$ with the line $y = k$. There exists a constant $C > 0$ such that for all $l \in \mathbb{N}$ large enough and for all $k \geq l$,*

$$\frac{\sup_{(p,q)\in\mathbb{A}} \#\left(Q_l(p,q) \cap \mathcal{B}_k\right)}{l^a} < C. \tag{16}$$

PROOF. Fix $l \in \mathbb{N}$. We compute

$$\sup_{k\geq l} \#(Q_l(w_k e^{l-k}, (k-l)e^l) \cap \mathcal{B}_k).$$

There are two cases to consider: $e^{2l}(k-l) > k+l$ and $e^{2l}(k-l) \leq k+l$. (Geometrically, these conditions correspond to whether the line $y = k$ hits the bottom edge of Q_l or the right edge of Q_l, respectively.) However, since we may assume that l is integer valued, there exists an integer L_0 such that for $l \geq L_0$, $e^{2l}(k-l) > k+l$ is always true for integers $k \geq l+1$. In this case, the largest possible x value in $Q_l(w_k e^{l-k}, (k-l)e^l) \cap \mathcal{B}_k$ is given by

$$x = \frac{(k+l)w_k}{(k-l)e^k}.$$

Hence, we need to estimate

$$\frac{w_k}{e^k}\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)^{n^{\frac{2}{a}}} \leq \frac{(k+l)w_k}{(k-l)e^k}.$$

For large l we obtain

$$n^{\frac{2}{a}} \leq \frac{\ln\left(\frac{k+l}{k-l}\right)}{\ln\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)} \leq 2l(l+1). \tag{17}$$

Thus

$$\sup_{k>l} \#(Q_l(w_k e^{l-k}, (k-l)e^l) \cap \mathcal{B}_k) = \left(\sup_{k>l}\left\lfloor \frac{\ln\left(\frac{k+l}{k-l}\right)}{\ln\left(\frac{k+\frac{1}{2}}{k-\frac{1}{2}}\right)}\right\rfloor\right)^{\frac{a}{2}} \leq \Big(2l(l+1)\Big)^{\frac{a}{2}} \tag{18}$$

for large l.

In the other case, $k = l$, instead of (17) we obtain an estimate

$$n^{\frac{2}{a}} \leq \frac{2l}{\ln\left(\frac{l+\frac{1}{2}}{l-\frac{1}{2}}\right)} = N_l.$$

We may now combine this with our earlier observation that $N_l \sim l^2$ for large l (cf., (15)) and with (18), to obtain the claimed inequality (16). □

The next result, Theorem 17, proves the existence of wave packet frames with dimensions $a \in (1,2]$. For this purpose we define the set $\mathcal{B} = \mathcal{B}(a) = \mathcal{B}_0 \cup (\{1\} \times \mathbb{Z})$, where $\mathcal{B}_0$ was defined in (11).

Theorem 17 *Let* $1 < a \leq 2$ *and* $\psi = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$. *Then the set* $\mathcal{B} = \mathcal{B}_0 \cup (\{1\} \times \mathbb{Z})$ *has dimension* $\dim^+(\mathcal{B}) = a$ *and* $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ *is a frame for* $L^2(\mathbb{R})$.

PROOF. Let $1 < a \leq 2$ be fixed. We start with an observation that the system $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is a frame for $L^2(\mathbb{R})$ (with frame bounds $A = 1$, $B = 2$). This follows from the fact that $\mathcal{WP}(\psi, \{1\} \times \mathbb{Z}^2)$ is an orthonormal basis for $L^2(\mathbb{R})$ and that $\mathcal{WP}(\psi, \mathcal{B}_0 \times \mathbb{Z})$ is a Bessel system (with Bessel constant 1), which in turn is a consequence of Lemma 15.

It remains to prove that $\dim^+(\mathcal{B}_0) = a$.

Step 1. First, we will show that $\dim^+(\mathcal{B}_0) \geq a$. We compute

$$\begin{aligned}
\limsup_{h\to\infty} \sup_{(p,q)\in\mathbb{A}} \frac{\#(Q_h(p,q) \cap \mathcal{B}_0)}{h^a} &\geq \limsup_{k\to\infty} \frac{\#(Q_k(w_k,0)) \cap \mathcal{B}_0)}{k^a} \\
&\geq \limsup_{k\to\infty} \frac{\#\left\{n \in \mathbb{Z} : 0 \leq n^{\frac{2}{a}} \leq N_k\right\}}{k^a} \\
&= \limsup_{k\to\infty} \frac{N_k^{\frac{a}{2}}}{k^a} = \lim_{k\to\infty} \frac{(k^2)^{\frac{a}{2}}}{k^a} = 1.
\end{aligned}$$

This implies $\dim^+(\mathcal{B}_0) \geq a$.

Step 2. The most difficult part of the proof is showing that $\dim^+(\mathcal{B}_0) \leq a$. Note that for the extreme case $a = 2$, this follows immediately from Theorem 3. Let, as before, $\mathcal{B}_k$ denote the intersection of $\mathcal{B}_0$ with the line $y = k$. Note that $\mathcal{B}_0 = \cup_{k \geq K} \mathcal{B}_k$. We begin by fixing an integer l and getting an upper estimate on $\#(Q_l(p,q) \cap \mathcal{B}_k)$ for any $k \geq l$. A first observation is that the equations of the top and bottom lines of $Q_l(p,q)$ are $y = \frac{q}{p}x \pm l$. So, by increasing p and/or decreasing q to 0, we decrease the slopes of the top and bottom lines. Therefore, in order to get an upper estimate, we may assume that $p \geq w_k e^{l-k}$. In fact, since $\mathcal{B}_k$ is most concentrated at w_k/e^k, we may assume that $p = w_k e^{l-k}$. Finally, by decreasing q and/or replacing (p,q) with (cp, cq), we may assume that $q = (k-l)e^l$, which positions the leftmost element of $\mathcal{B}_k$ at the upper-left corner of $Q_l(p,q)$ and thus maximizes the length of $\mathcal{B}_k \cap Q_l(p,q)$. We omit the routine (but tedious) formal verification of the above reductions.

Next we note that, due to our construction, the set $\mathcal{B}_0$ has the following property: if l is large enough and if there exists $k \geq l$ such that the set

$Q_l(p,q)\cap\mathcal{B}_k$ is non-empty then, for all $j\neq k$ (including $j<k$), $Q_l(p,q)\cap\mathcal{B}_j=\emptyset$. (Here p and q are arbitrary.)

This last observation allows us, in particular, to rewrite (16) of Lemma 16, as

$$\limsup_{l\to\infty}\frac{\sup_{(p,q)\in\mathbb{A}}\#\left(Q_l(p,q)\cap\bigcup_{k\geq l}\mathcal{B}_k\right)}{l^a}<\infty, \tag{19}$$

and, in view of (19), this also means that now we only need to estimate $\sup_{(p,q)\in\mathbb{A}}\#(Q_l(p,q)\cap\mathcal{B}_k)$ for $k<l$, when l is large enough.

The next step in our proof is the observation that in order to show that $\dim^+(\mathcal{B}_0)\leq a$, it is enough to obtain the following estimate for each $\epsilon>0$:

$$\limsup_{l\to\infty}\frac{\sup_{(p,q)\in\mathbb{A}}\#(Q_l(p,q)\cap\mathcal{B}_0)}{l^{a+\epsilon}}<\infty. \tag{20}$$

Consider now the case $k<l$. First we ask — how many k's are there such that $\mathcal{B}_k\subset Q_l(p,q)$? If the left edge of $Q_l(p,q)$ is not at the leftmost point of some set $\mathcal{B}_k$, we may move the edge to the right so that the left endpoint of the first $\mathcal{B}_k$ contained in $Q_l(p,q)$ is the same as the left edge of $Q_l(p,q)$. This can allow us to include more sets $\mathcal{B}_k$ in $Q_l(p,q)$, but certainly not fewer. Thus, we may assume that $pe^{-l}=w_je^{-j}$ for some j (recall that $w_k=e^{e^k+k}$). From the construction of the set $\mathcal{B}_0$ (see the choice of K in the proof of Lemma 15) it follows that in this case $Q_l(p,q)\cap\mathcal{B}_k=\emptyset$ for all $k<j$. On the other hand for $k>\ln\ln(w_je^{2l-j})$, we have that $\frac{w_k}{e^k}\geq pe^l$, and so $Q_l(p,q)\cap\mathcal{B}_k=\emptyset$. Moreover,

$$\ln\ln(w_je^{2l-j})\leq j+\ln(2l+1).$$

Now, we notice that the number of k's for which $\mathcal{B}_k\cap Q_l(p,q)\neq\emptyset$ is no more than two plus the number of k's such that $\mathcal{B}_k\subset Q_l(p,q)$, which follows from the fact that we were estimating the x-values of the sets.

So, for fixed, large enough, l, we have that, for $\epsilon>0$,

$$\begin{aligned}&\limsup_{l\to\infty}\frac{\sup_{(p,q)\in\mathbb{A}}\#(Q_l(p,q)\cap\bigcup_{k<l}\mathcal{B}_k)}{l^{a+\epsilon}}\\ &\leq\limsup_{l\to\infty}\frac{1}{l^{a+\epsilon}}\sup_{j\leq l}\sum_{i=j-1}^{j+\ln(2l+1)+1}\left(2i(i+\frac{1}{2})\right)^{\frac{a}{2}}\\ &\leq\limsup_{l\to\infty}\frac{3+\ln(2l+1)}{l^{a+\epsilon}}\left(2(l+\ln(2l+1)+1)(l+\ln(2l+1)+\frac{3}{2})\right)^{\frac{a}{2}}=0.\end{aligned}$$

Here, we used (15) and the observation preceding (19).

Now, combining the said observation with the above calculation and with (19), we obtain (20), and so $\dim^+(\mathcal{B}_0) \leq a$, as desired. □

Remark 18 *If one is more careful in the construction of* $\mathcal{B}_0$*, one can obtain a set* $\mathcal{B}$ *such that* $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *and* $\dim^+(\mathcal{B}) = 2$*. Since this is even more complicated than the above argument, and we are unable to generalize it to* $1 < \dim^+(\mathcal{B}) < 2$*, we choose not to include the argument here.*

We are now able to obtain a full description of which values the upper and lower dimension associated with a wave packet frame can attain in the case $\mathcal{M} = \mathcal{B} \times \mathbb{Z}$ under consideration.

Theorem 19 *(1) For each* $0 \leq a \leq 2$*, there exists a* $\psi \in L^2(\mathbb{R})$ *and* $\mathcal{B} \subset \mathbb{A}$ *with* $\dim^+(\mathcal{B}) = a$ *such that* $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ *is a frame for* $L^2(\mathbb{R})$*. Moreover, these are all possible values.*

(2) If there exists a $\psi \in L^2(\mathbb{R})$ *and* $\mathcal{B} \subset \mathbb{A}$ *such that* $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ *is a frame for* $L^2(\mathbb{R})$*, then* $\dim^-(\mathcal{B}) = 0$*.*

PROOF. (1): For $0 < a \leq 2$, this follows from Theorems 14 and 17.

Next we study the case $a = 0$. For this, consider the sequence $(x_n)_{n \in \mathbb{N}}$ defined by $x_n = e^{-e^n}$, and let $h, n, x > 0$. Then we obtain

$$x_n \in [xe^{-h}, xe^h] \iff -\ln x - h \leq e^n \leq -\ln x + h.$$

For each $h > 3$, this yields

$$\begin{aligned}\max_{y>0} \#(n \in \mathbb{N}_0 : e^n \in [-\ln y - h, -\ln y + h]) &\leq \#(n \in \mathbb{N} : e^n \in [1, 2h+1]) \\ &\leq 1 + \ln(2h+1).\end{aligned}$$

It is an easy calculation to check that $(x_n)_{n \in \mathbb{N}}$ satisfies the conditions of Proposition 13. Hence there exists a bounded sequence $(y_n)_{n \in \mathbb{N}}$ such that the set $\mathcal{WP}(\mathbf{1}_{[1,2]}, \{(x_n, y_n) : n \in \mathbb{N}\} \times \mathbb{Z})$ is an orthonormal basis. Now we choose $\psi = \mathbf{1}_{[1,2]}$ and $\mathcal{B} = \{(x_n, y_n) : n \in \mathbb{N}\}$. It remains to prove that $\dim^+(\mathcal{B}) = 0$. For each $A > 0$, using the boundedness of $(y_n)_{n \in \mathbb{N}}$, we compute

$$\begin{aligned}\mathcal{D}_A^+(\mathcal{B}) &= \limsup_{h \to \infty} \sup_{(x,y) \in \mathbb{A}} \frac{\#(\mathcal{B} \cap Q_h(x,y))}{h^A} \\ &= \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^+} \frac{\#(\{n \in \mathbb{N}_0 : x_n \in [xe^{-h}, xe^h])}{h^A} \\ &\leq \limsup_{h \to \infty} \frac{1 + \ln(2h+1)}{h^A} = 0.\end{aligned}$$

Thus $\dim^+(\mathcal{B}) = 0$.

By Theorem 9, the only values which $\dim^+(\mathcal{B})$ can attain are elements of $[0, 2] \cup \{\infty\}$. So it remains to deal with the case $a = \infty$.

We claim that for all $\mathcal{B} \subset \mathbb{A}$ with $\dim^+(\mathcal{B}) = \infty$ and for all functions $\psi \in L^2(\mathbb{R})$, the set $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is not a frame for $L^2(\mathbb{R})$. Towards a contradiction, we assume there exist some $\mathcal{B} \subset \mathbb{A}$ with $\dim^+(\mathcal{B}) = \infty$ and a function $\psi \in L^2(\mathbb{R})$ such that the set $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is a Bessel system. By definition, $\dim^+(\mathcal{B}) = \infty$ implies that $\mathcal{D}_1^+(\mathcal{B}) = \infty$. However, by Theorem 3, $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ can only be a Bessel system if $\mathcal{D}_2^+(\mathcal{B} \times \mathbb{Z}) < \infty$. Using Lemma 8, this implies $\mathcal{D}_1^+(\mathcal{B}) < \infty$, a contradiction.

(2): Let $\mathcal{B} \subset \mathbb{A}$ be such that there exists a wave packet frame for $\mathcal{B} \times \mathbb{Z}$. By Theorem 10, we have $\mathcal{D}_2^-(\mathcal{B}) = 0$. Hence, by Corollary 7, $\mathcal{D}_A^-(\mathcal{B}) = 0$ for all $0 < A < \infty$. This implies that $\dim^-(\mathcal{B}) = 0$. □

The last theorem is a summary of the main results in this section.

Theorem 20 *Let $\psi \in L^2(\mathbb{R})$ and $\mathcal{B}$ be a discrete subset of $\mathbb{A}$.*

(1) If $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is a frame for $L^2(\mathbb{R})$, then $\dim^-(\mathcal{B} \times \mathbb{Z}) = 0$, and $\dim^+(\mathcal{B} \times \mathbb{Z}) \leq 3$.
(2) For each $1 \leq a \leq 3$, there exists a function $\psi \in L^2(\mathbb{R})$ and a set $\mathcal{B} \subset \mathbb{A}$ such that $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ is a frame and $\dim^+(\mathcal{B} \times \mathbb{Z}) = a$.

PROOF. For part (1), let $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ be a frame for $L^2(\mathbb{R})$. Then, by Theorem 19 (2), $\mathcal{D}_A^-(\mathcal{B}) = 0$ for all $0 < A < \infty$. Now Lemma 8 implies that $\mathcal{D}_A^-(\mathcal{B} \times \mathbb{Z}) = 0$ for all $1 < A < \infty$, and hence $\dim^-(\mathcal{B} \times \mathbb{Z}) \leq 1$. But this can only happen if $\dim^-(\mathcal{B} \times \mathbb{Z}) = 0$ by Theorem 2.

Concerning the upper dimension, again by Theorem 2, it suffices to prove that $\dim^+(\mathcal{B} \times \mathbb{Z}) < \infty$. But this follows immediately from Theorem 19 (1) and Lemma 8.

Part (2) is an immediate application of Theorem 17, Theorem 14 and Lemma 8. □

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